

WACH MODULES AND IWASAWA THEORY FOR MODULAR FORMS

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ABSTRACT. We define a family of Coleman maps for positive crystalline p -adic representations of the absolute Galois group of \mathbb{Q}_p using the theory of Wach modules. Let $f = \sum a_n q^n$ be a normalized new eigenform and p an odd prime at which f is either good ordinary or supersingular. By applying our theory to the p -adic representation associated to f , we define Coleman maps $\underline{\text{Col}}_i$ for $i = 1, 2$ with values in $\overline{\mathbb{Q}_p} \otimes_{\mathbb{Z}_p} \Lambda$, where Λ is the Iwasawa algebra of \mathbb{Z}_p^\times . Applying these maps to the Kato zeta elements gives a decomposition of the (generally unbounded) p -adic L -functions of f into linear combinations of two power series of bounded coefficients, generalizing works of Pollack (in the case $a_p = 0$) and Sprung (when f corresponds to a supersingular elliptic curve). Using ideas of Kobayashi for elliptic curves which are supersingular at p , we associate to each of these power series a Λ -cotorsion Selmer group. This allows us to formulate a “main conjecture”. Under some technical conditions, we prove one inclusion of the “main conjecture” and show that the reverse inclusion is equivalent to Kato’s main conjecture.

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1. INTRODUCTION

1.1. Background. Let E be an elliptic curve defined over \mathbb{Q} which has good ordinary reduction at the prime p . In [MSD74], Mazur and Swinnerton-Dyer constructed a p -adic L -function, $\tilde{L}_{p,E}$, which interpolates complex L -values of E . Let $\mathbb{Q}_\infty = \mathbb{Q}(\mu_{p^\infty})$. If G_∞ denotes the Galois group of \mathbb{Q}_∞ over \mathbb{Q} , then $\tilde{L}_{p,E}$ is an element of $\Lambda_{\mathbb{Q}_p}(G_\infty) = \mathbb{Q} \otimes \mathbb{Z}_p[[G_\infty]]$. It is conjectured that $\tilde{L}_{p,E}$ is in fact an element of the Iwasawa algebra $\Lambda(G_\infty) = \mathbb{Z}_p[[G_\infty]]$.

Recall that the p -Selmer group of E over any finite extension F of \mathbb{Q} is defined as

$$\text{Sel}_p(E/F) = \ker \left(H^1(F, E_{p^\infty}) \longrightarrow \prod_v \frac{H^1(F_v, E_{p^\infty})}{E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right),$$

where the product is taken over all places of F . If we let $\text{Sel}_p(E/\mathbb{Q}_\infty) = \varinjlim_n \text{Sel}_p(E/\mathbb{Q}(\mu_{p^n}))$, then $\text{Sel}_p(E/\mathbb{Q}_\infty)$ is equipped with an action of G_∞ which extends to an action of the Iwasawa algebra. It is not difficult to show that the Pontryagin dual $\text{Sel}_p(E/\mathbb{Q}_\infty)^\vee$ is finitely generated over $\Lambda(G_\infty)$, and a theorem of Kato-Rohrlich (conjectured by Mazur) states that it is in fact $\Lambda(G_\infty)$ -torsion. We can therefore associate to it a characteristic ideal for each Δ -isotypical component, where Δ is the torsion subgroup of G_∞ , and the main conjecture of cyclotomic Iwasawa theory for E predicts that this ideal is generated by the corresponding isotypical component of $\tilde{L}_{p,E}$.

The construction of p -adic L -functions has been generalized to more general primes and modular forms in [AV75, Viš76]. If $f = \sum a_n q^n$ is a normalized new eigenform of weight $k \geq 2$, level N and character ϵ , $p \nmid N$, then there exists a p -adic L -function $\tilde{L}_{p,\alpha}$, for any root α of $X^2 - a_p X + \epsilon(p)p^{k-1}$ such that $v_p(\alpha) < k-1$, interpolating complex L -values of f . Perrin-Riou [PR95] and Kato [Kat93] have established theories of p -adic L -functions for a wide class of p -adic Galois representations and formulated respective Iwasawa main conjectures. When the representation corresponds to a modular form, these main conjectures have been reformulated by Kato [Kat04] using the theory of Euler systems. If f is good ordinary at p (in other words, $p \nmid N$ and a_p is a p -adic unit) and α is the unique unit root, then $\tilde{L}_{p,\alpha}$ is an element of $\Lambda_{\mathbb{Q}_p}(G_\infty)$. At a Δ -isotypical component, the main conjecture is equivalent to asserting that $\tilde{L}_{p,\alpha}$ generates the characteristic ideal of $\text{Sel}_p(f/\mathbb{Q}_\infty)^\vee$. In *op.cit.*, Kato has shown that $\tilde{L}_{p,\alpha}$ is contained in the characteristic ideal of $\text{Sel}_p(f/\mathbb{Q}_\infty)^\vee$ under some technical assumptions; his proof relies on the construction of certain zeta elements (which we will refer to as *Kato zeta elements*).

When f is supersingular at p (by which we mean $p \nmid N$ and a_p is not a p -adic unit), two problems arise: on the one hand, the p -adic L -functions of Amice-Vélu and Vishik are no longer elements of $\Lambda(G_\infty)$, but they lie in the algebra $\mathcal{H}(G_\infty)$ of distributions on G_∞ , and on the other hand, $\text{Sel}_p(f/\mathbb{Q}_\infty)^\vee$ is no longer $\Lambda(G_\infty)$ -torsion. Perrin-Riou's (and hence Kato's) main conjecture can therefore not be translated into a statement relating $\tilde{L}_{p,\alpha}$ and $\text{Sel}_p(f/\mathbb{Q}_\infty)$ as in the ordinary case. When $a_p = 0$, a remedy was made possible by the works of Pollack [Pol03]. If α_1 and α_2 are the roots of $X^2 + \epsilon(p)p^{k-1}$, Pollack showed that there is a decomposition

$$\tilde{L}_{p,\alpha_i} = \log_{p,k}^+ \tilde{L}_p^+ + \alpha_i \log_{p,k}^- \tilde{L}_p^-$$

for $i = 1, 2$, where $\tilde{L}_p^\pm \in \Lambda_{\mathbb{Q}_p}(G_\infty)$ and $\log_{p,k}^\pm$ are some explicit elements of $\mathcal{H}(G_\infty)$ which only depend on k . When f corresponds to an elliptic curve E/\mathbb{Q} (and $p > 2$), the \tilde{L}_p^\pm are in fact elements of $\Lambda(G_\infty)$. In [Kob03], Kobayashi formulates a main conjecture giving an arithmetic interpretation of these new p -adic

L -functions. In analogy to the ordinary reduction case, he defines even and odd Selmer groups $\text{Sel}_p^\pm(E/\mathbb{Q}_\infty)$ by modifying the local conditions at p in the definition of the usual Selmer group. Let $T_p E$ be the Tate module of E . Kobayashi shows that $\text{Sel}_p^\pm(E/\mathbb{Q}_\infty)$ is $\Lambda(G_\infty)$ -cotorsion by constructing the so-called plus and minus Coleman maps

$$\text{Col}^\pm : H_{\text{Iw}}^1(\mathbb{Q}_p, T_p E) \rightarrow \Lambda(G_\infty),$$

which depend on the structure of the formal group attached to E . (Here $H_{\text{Iw}}^1(\mathbb{Q}_p, T_p E)$ is the Iwasawa cohomology, defined as $\varprojlim_n H^1(\mathbb{Q}_p(\mu_{p^n}), T_p E)$; see §2.3 below.) Kobayashi's modified main conjecture then asserts that in each Δ -isotypical component, the functions \tilde{L}_p^\pm generate the respective characteristic ideals of $\text{Sel}_p^\pm(E/\mathbb{Q}_\infty)^\vee$. This main conjecture is in fact equivalent to [Kat04, Conjecture 12.10] (to which we refer as *Kato's main conjecture* from now on). Using the fact that the maps Col^\pm send the localization of the Kato zeta elements to \tilde{L}_p^\pm , Kobayashi shows that the elements \tilde{L}_p^\pm are contained in the characteristic ideals of $\text{Sel}_p^\pm(E/\mathbb{Q}_\infty)^\vee$ (possibly after inverting p if p is one of the finitely many primes for which the p -adic Galois representation of E is not surjective), establishing half of the main conjecture. (When the elliptic curve has complex multiplication, the full conjecture has been proved by Pollack and Rubin [PoR04].)

Sprung [Spr09] has extended the results of Kobayashi to elliptic curves with supersingular reduction at p and $a_p \neq 0$ (which forces p to be 2 or 3). He constructs Coleman maps

$$\text{Col}^\vartheta, \text{Col}^v : H_{\text{Iw}}^1(\mathbb{Q}_p, T_p E) \longrightarrow \Lambda(G_\infty)$$

and defines $\tilde{L}_p^\vartheta, \tilde{L}_p^v \in \Lambda(G_\infty)$ by applying these Coleman maps to the Kato zeta element. Analogously to the case $a_p = 0$ discussed above, he defines two Selmer groups $\text{Sel}_p^\vartheta(E/\mathbb{Q}_\infty)$ and $\text{Sel}_p^v(E/\mathbb{Q}_\infty)$ to formulate the corresponding main conjectures. Moreover, he constructs a matrix $M \in M_2(\mathcal{H}(G_\infty))$ whose entries are functions of logarithmic growth depending only on a_p such that

$$\begin{pmatrix} \tilde{L}_{p,\alpha} \\ \tilde{L}_{p,\beta} \end{pmatrix} = M \begin{pmatrix} \tilde{L}_p^\vartheta \\ \tilde{L}_p^v \end{pmatrix}$$

generalizing Pollack's results.

Generalizing Kobayashi's work in a different direction, the first author has shown in [Lei09] that the definition of the maps Col^\pm can be extended to general modular forms with $a_p = 0$, using p -adic Hodge theory in place of formal groups. For a normalized new eigenform f , there exists a p -adic representation V_f of $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to f , as constructed by Deligne [Del69]. When $a_p = 0$, one can then construct \pm -Coleman maps

$$\text{Col}^\pm : H_{\text{Iw}}^1(\mathbb{Q}_p, V_f) \rightarrow \Lambda_{\mathbb{Q}_p}(G_\infty),$$

using the structure of $\mathbb{D}_{\text{cris}}(V_f)$ and Perrin-Riou's exponential map (see Section 2 in [Lei09]). Generalizing Kobayashi's construction, one can use Col^\pm to define \pm -Selmer groups, which again turn out to be $\Lambda(G_\infty)$ -cotorsion and whose characteristic ideals at each Δ -isotypical component contain Pollack's p -adic L -functions. Analogous to the work of Pollack and Rubin for elliptic curves, one can show that equality holds for forms of CM type; see [Lei09] for details.

1.2. Statement of the main results. Looking at all these results raises some natural questions: Is there a uniform explanation for Sprung's logarithmic matrix M and Pollack's \pm -logarithms? Can one generalize the construction of the two Coleman series to more general modular forms which are supersingular at p ?

In this paper, we approach these questions using methods from the theory of (φ, G_∞) -modules. As shown by Fontaine (unpublished – for a reference see [CC99]), for any \mathbb{Z}_p -linear representation T of $G_{\mathbb{Q}_p}$ there is a canonical isomorphism $h_{\mathbb{Q}_p, \text{Iw}}^1 : H_{\text{Iw}}^1(\mathbb{Q}_p, T) \cong \mathbb{D}(T)^{\psi=1}$, where $\mathbb{D}(T)$ denotes the (φ, G_∞) -module¹ of T and ψ is a certain left inverse of φ . Recall that $\mathbb{D}(T)$ is a module over the p -adic completion $\mathbb{A}_{\mathbb{Q}_p}$ of the power series ring $\mathbb{Z}_p[[\pi]][\pi^{-1}]$. Also, $\Lambda(G_\infty)$ can be identified with the additive group $\mathbb{Z}_p[[\pi]]^{\psi=0}$ via the Mellin transform (c.f. Section 5.1). It seems therefore natural to expect that by carefully choosing a basis of $\mathbb{D}(T)$, it should be possible to define the two Coleman maps as certain maps on the coefficients of an element

¹More familiarly known as a (φ, Γ) -module – our G_∞ is denoted by Γ in Fontaine's work, while we use Γ for its torsion-free part.

$x \in \mathbb{D}(T)^{\psi=1}$. Such a construction would generalize the classical case $T = \mathbb{Z}_p(1)$: in this case, the Coleman map $H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \cong \mathbb{A}_{\mathbb{Q}_p}^{\psi=1} \rightarrow \mathbb{Z}_p[[\pi]][\pi^{-1}]^{\psi=0}$ is just the map $\varphi - 1$.

Here, we develop this idea using Berger's theory of Wach modules [Ber03], which is a refined version of (φ, G_∞) -modules for crystalline representations over unramified base fields originally studied by Wach in [Wac96]. The Wach module $\mathbb{N}(V)$ of a crystalline $G_{\mathbb{Q}_p}$ -representation V is a certain subspace of the (φ, G_∞) -module $\mathbb{D}(V)$ which is a finitely-generated module over the simpler ring $\mathbb{B}_{\mathbb{Q}_p}^+ = \mathbb{Z}_p[[\pi]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If V is a crystalline representation of $G_{\mathbb{Q}_p}$ with non-negative Hodge-Tate weights, and V has no quotient isomorphic to \mathbb{Q}_p , then Berger has shown in [Ber03] that $\mathbb{D}(V)^{\psi=1} = \mathbb{N}(V)^{\psi=1}$. Let $\varphi^* \mathbb{N}(V)$ be the $\mathbb{B}_{\mathbb{Q}_p}^+$ -submodule of $\mathbb{D}(V)$ generated by the image of φ . For any such representation, $1 - \varphi$ gives a map

$$1 - \varphi : \mathbb{N}(V)^{\psi=1} \longrightarrow (\varphi^* \mathbb{N}(V))^{\psi=0}.$$

Our first main result relates this map to Perrin-Riou's theory. Suppose that V_f is the p -adic representation associated to a modular form f with p a good prime for f , i.e. p does not divide the level of f (we assume here for notational simplicity that the coefficient field of the modular form is \mathbb{Q} , so V is a 2-dimensional \mathbb{Q}_p -vector space). Let $V = V_f(k-1)$, then V is a crystalline representation with Hodge-Tate weights $0, k-1$. We fix $\bar{\nu}_1, \bar{\nu}_2$ a basis of $\mathbb{D}_{\text{cris}}(V_f)$ in Section 3.3. It lifts to a basis n_1, n_2 of $\mathbb{N}(V_f)$. Note that $\pi^{1-k} n_1 \otimes e_{k-1}, \pi^{1-k} n_2 \otimes e_{k-1}$ then gives a basis of $\mathbb{N}(V)$. Let $M = (m_{ij}) \in M_2(\varphi(\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+))$ be such that

$$\begin{pmatrix} \varphi(\pi^{1-k} n_1 \otimes e_{k-1}) \\ \varphi(\pi^{1-k} n_2 \otimes e_{k-1}) \end{pmatrix} = M \begin{pmatrix} \bar{\nu}_1 \otimes t^{1-k} e_{k-1} \\ \bar{\nu}_2 \otimes t^{1-k} e_{k-1} \end{pmatrix}.$$

Proposition 1.1 (see Proposition 3.22). *For $i = 1, 2$ we have a commutative diagram*

$$\begin{array}{ccc} \mathbb{N}(V)^{\psi=1} & \xrightarrow{h_{\mathbb{Q}_p, \text{Iw}}^1} & H_{\text{Iw}}^1(\mathbb{Q}_p, V) \\ \downarrow 1-\varphi & & \downarrow \\ (\varphi^* \mathbb{N}(V))^{\psi=0} & & \\ \downarrow M & & \downarrow \mathcal{L}_{1, \bar{\nu}_i \otimes (1+\pi)} \\ ((\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0})^{\oplus 2} & & \\ \downarrow \text{pr}_i & & \downarrow \\ (\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0} & \xrightarrow{\mathfrak{M}^{-1}} & \mathcal{H}(G_\infty) \end{array}$$

Here, $\mathcal{L}_{1, \bar{\nu}_i \otimes (1+\pi)}$ is a certain $\Lambda_{\mathbb{Q}_p}$ -module homomorphism whose definition is given in equation (18) below, defined using Perrin-Riou's exponential map and the Perrin-Riou pairing $H_{\text{Iw}}^1(V) \times H_{\text{Iw}}^1(V^*(1)) \rightarrow \Lambda_{\mathbb{Q}_p}$. Also, \mathfrak{M} is the inverse Mellin transform (see (13)), pr_i is the projection map onto the i -th component, and for an element $x \in (\varphi^* \mathbb{N}(V))^{\psi=0}$, $M.x$ is defined as follows: if $x = x_1 \varphi(\pi^{1-k} n_1 \otimes e_{k-1}) + x_2 \varphi(\pi^{1-k} n_2 \otimes e_{k-1})$ for some $x_i \in (\mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$, then $M.x = M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

By applying this diagram to Kato's zeta element $\mathbf{z}^{\text{Kato}} \in H_{\text{Iw}}^1(\mathbb{Q}_p, V)$, we deduce that there exist $\mathcal{M} \in M_2(\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \varphi(\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+))$ and $L_{p,1}, L_{p,2} \in (\mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$ (c.f. Section 3.5.1), depending only on the basis n_1, n_2 , such that we have a decomposition

$$(1) \quad \begin{pmatrix} \mathfrak{M}(\tilde{L}_{p,\alpha}) \\ \mathfrak{M}(\tilde{L}_{p,\beta}) \end{pmatrix} = \mathcal{M} \begin{pmatrix} L_{p,1} \\ L_{p,2} \end{pmatrix}.$$

In order to interpret this decomposition in terms of measures, we need to study the structure of $(\varphi^* \mathbb{N}(V))^{\psi=0}$ as a $\Lambda(G_\infty)$ -module. The following result was proven independently by Berger (Theorem 3.5, for general Wach modules) and ourselves (Theorem 4.24, for the Wach module of the representation arising from a supersingular modular form).

Theorem 1.2. *Let \mathcal{N} be a Wach module of rank d . Then $(\varphi^* \mathcal{N})^{\psi=0}$ is a free $\Lambda_{\mathbb{Q}_p}(G_\infty)$ -module of rank d . Moreover, there exists a basis n_1, \dots, n_d of \mathcal{N} such that $(1 + \pi)\varphi(n_1), \dots, (1 + \pi)\varphi(n_d)$ is a $\Lambda_{\mathbb{Q}_p}(G_\infty)$ -basis of $(\varphi^* \mathbb{N}(V))^{\psi=0}$.*

When V is the p -adic representation associated to a modular form with $v_p(a_p) > \lfloor \frac{k-2}{p-1} \rfloor$, then there is an explicit choice of the $\mathbb{B}_{\mathbb{Q}_p}^+$ -basis of $\mathbb{N}(V)$ which was constructed in [BLZ04] (c.f. Section 4). We show (Theorem 4.24) that this basis (n_1, n_2) has the additional property that $(1 + \pi)\varphi(n_1), (1 + \pi)\varphi(n_2)$ is a $\Lambda_{\mathbb{Q}_p}(G_\infty)$ -basis of $(\varphi^* \mathbb{N}(V))^{\psi=0}$. Hence we may define the *Iwasawa transform*

$$\mathfrak{J} : (\varphi^* \mathbb{N}(V))^{\psi=0} \longrightarrow \Lambda_{\mathbb{Q}_p}(G_\infty)^{\oplus 2}$$

to be the induced isomorphism of $\Lambda_{\mathbb{Q}_p}(G_\infty)$ -modules associated to this basis. This map has the following property: if $a_p = 0$, then \mathfrak{J} fits into the commutative diagram

$$\begin{array}{ccc} \mathbb{N}(V)^{\psi=1} & \xrightarrow{h_{\mathbb{Q}_p, \text{Iw}}^1} & H_{\text{Iw}}^1(\mathbb{Q}_p, V) \\ \downarrow 1-\varphi & & \downarrow (\text{Col}^\pm) \\ (\varphi^* \mathbb{N}(V))^{\psi=0} & \xrightarrow{\mathfrak{J}} & \Lambda_{\mathbb{Q}_p}(G_\infty)^{\oplus 2} \end{array}$$

where Col^\pm are the Coleman maps constructed in [Kob03] and [Lei09]. In other words, if $i = 1, 2$ and we define $\underline{\text{Col}}_i : \mathbb{N}(V)^{\psi=1} \rightarrow \Lambda_{\mathbb{Q}_p}(G_\infty)$ to be the composition of $\mathfrak{J} \circ (1 - \varphi)$ with the projection of $\Lambda_{\mathbb{Q}_p}(G_\infty)^{\oplus 2}$ onto the i -th component, then we recover the constructions in *op.cit*. In Sections 5.2 and 5.3, we use this new description of the Coleman maps to give alternative proofs of their main properties.

When $v_p(a_p) > \lfloor \frac{k-2}{p-1} \rfloor$, we define the maps $\underline{\text{Col}}_i$ in the same manner, and it follows from Proposition 3.24 that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{N}(V)^{\psi=1} & \xrightarrow{h_{\mathbb{Q}_p, \text{Iw}}^1} & H_{\text{Iw}}^1(\mathbb{Q}_p, V) \\ \downarrow 1-\varphi & & \downarrow (\underline{\text{Col}}_1, \underline{\text{Col}}_2) \\ (\varphi^* \mathbb{N}(V))^{\psi=0} & \xrightarrow{\mathfrak{J}} & \Lambda_{\mathbb{Q}_p}(G_\infty)^{\oplus 2} \\ \downarrow M & & \downarrow \underline{M} \\ ((\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0})^{\oplus 2} & \xrightarrow{\mathfrak{M}^{-1}} & \mathcal{H}(G_\infty)^{\oplus 2} \\ \downarrow \text{pr}_i & & \downarrow \underline{\text{pr}}_i \\ (\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0} & \xrightarrow{\mathfrak{M}^{-1}} & \mathcal{H}(G_\infty). \end{array}$$

$\mathcal{L}_{1, \nu_i \otimes (1 + \pi)}$

Here, the map $\underline{\text{pr}}_i$ is the projection onto the i -th component in $\mathcal{H}(G_\infty)^{\oplus 2}$. In particular, this diagram allows us to translate (1) in terms of $\Lambda_{\mathbb{Q}_p}(G_\infty)$:

Theorem 1.3 (see Theorem 3.25). *For $i = 1, 2$, define $\tilde{L}_{p,i} = \underline{\text{Col}}_i(\mathbf{z}^{\text{Kato}})$. There exists a 2×2 -matrix $\underline{\mathcal{M}} \in M_2(\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathcal{H}(G_\infty))$ depending only on k and a_p such that*

$$(2) \quad \begin{pmatrix} \tilde{L}_{p,\alpha} \\ \tilde{L}_{p,\beta} \end{pmatrix} = \underline{\mathcal{M}} \begin{pmatrix} \tilde{L}_{p,1} \\ \tilde{L}_{p,2} \end{pmatrix}$$

We show in Proposition 5.10 that this decomposition reduces to the decompositions of $\tilde{L}_{p,\alpha}, \tilde{L}_{p,\beta}$ given by Pollack when $a_p = 0$.

Assume now that V_f is the p -adic representation associated to a modular form f which is good ordinary at p , and let $V = V_f(k-1)$. By choosing a suitable basis for $\mathbb{D}_{\text{cris}}(V_f)$ (c.f Section 3.6) and applying Theorem 3.5 to $(\varphi^* \mathbb{N}(V))^{\psi=0}$, we can proceed analogously to the supersingular case discussed above to construct Coleman maps $\underline{\text{Col}}_i : \mathbb{N}(V)^{\psi=1} \rightarrow \Lambda_{\mathbb{Q}_p}(G_\infty)$. Let α and β be the unit and non-unit eigenvalues of the Frobenius respectively. The Kato zeta element gives rise to two p -adic L -functions $\tilde{L}_{p,\alpha}$ and $\tilde{L}_{p,\beta}$, where $\tilde{L}_{p,\beta}$ conjecturally agrees with the critical-slope p -adic L -function constructed by Pollack and Stevens in [PoS09] when V_f is not locally split at p . The analogue of (2) becomes

$$(3) \quad \begin{pmatrix} \tilde{L}_{p,\alpha} \\ \tilde{L}_{p,\beta} \end{pmatrix} = \begin{pmatrix} 0 & \bar{u} \\ -\alpha \log_{p,k} & * \end{pmatrix} \begin{pmatrix} \tilde{L}_{p,1} \\ \tilde{L}_{p,2} \end{pmatrix}.$$

for some $\bar{u} \in \Lambda_E(G_\infty)^\times$ (c.f. (37)). Note that a similar decomposition can be obtained from works of Perrin-Riou for elliptic curves with good ordinary reduction at p (see [PR93, Section 1.4]). The decomposition (3) allows us to show that $\tilde{L}_{p,1}, \tilde{L}_{p,2} \neq 0$ under some technical assumptions.

As in the cases studied in [Kob03] and [Lei09], we can use the maps $\underline{\text{Col}}_i$ to construct Selmer groups $\text{Sel}_p^i(f/\mathbb{Q}_\infty)$ (see Definition 6.4), and we prove the following results. Define assumptions

- (A) (when f is supersingular at p) $k \geq 3$ or $a_p = 0$;
- (A') (when f is good ordinary at p) $k \geq 3$ and V_f is not locally split at p .

Theorem 1.4 (see Theorem 6.5). *Under assumption (A) (if f is supersingular at p) or assumption (A') (if f is good ordinary at p), the group $\text{Sel}_p^i(f/\mathbb{Q}_\infty)$ is $\Lambda_{\mathcal{O}_E}(G_\infty)$ -cotorsion for $i = 1, 2$. Moreover, there exist some $n_i \geq 0$ such that*

$$\varpi^{n_i} \tilde{L}_{p,i}^\eta \in \text{Char}_{\Lambda_{\mathcal{O}_E}(\Gamma)}(\text{Sel}_p^i(f/\mathbb{Q}_\infty)^{\vee, \eta})$$

where η is any character on Δ when $i = 1$ and it is the trivial character when $i = 2$.

Corollary 1.5 (see Corollary 6.6). *Let η be a character on Δ as in Theorem 1.4. If either assumption (A) or assumption (A') is satisfied, and the image of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_\infty)$ in $\text{GL}(V_f)$ contains a conjugate of $\text{SL}_2(\mathbb{Z}_p)$, then Kato's main conjecture is equivalent to*

$$\text{Char}_{\Lambda_{\mathcal{O}_E}(\Gamma)}(\text{Sel}_p^i(f/\mathbb{Q}_\infty)^{\vee, \eta}) = \text{Char}_{\Lambda_{\mathcal{O}_E}(\Gamma)}(\text{Im}(\underline{\text{Col}}_i)^\eta / (\tilde{L}_{p,i}^\eta)).$$

Note that in the ordinary case, $\tilde{L}_{p,2}$ agrees with the usual p -adic L -function of f up to a unit in $\Lambda_E(G_\infty)$. It will be shown in a forthcoming paper of the first and third authors [LZ10] that the corresponding Selmer group is the usual $\text{Sel}_p(f/\mathbb{Q}_\infty)$; whereas the first Coleman map gives a new p -adic L -function $\tilde{L}_{p,1}$ and a new Selmer group.

1.3. Notation. Throughout this paper, let p be an odd prime. Fix embeddings of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$, and into \mathbb{C} . For $n \geq 0$, write $\mathbb{Q}_{p,n} = \mathbb{Q}_p(\mu_{p^n})$ (resp. $\mathbb{Q}_n = \mathbb{Q}(\mu_{p^n})$) for the extension of \mathbb{Q}_p (resp. \mathbb{Q}) obtained by adjoining the p^n -th roots of unity. Let G_n denote its Galois group. Let $\mathbb{Q}_{p,\infty} = \bigcup \mathbb{Q}_{p,n}$, and write G_∞ for the Galois group of $\mathbb{Q}_{p,\infty}$ over \mathbb{Q}_p . We identify G_∞ with the Galois group of $\mathbb{Q}_\infty = \bigcup_{n \geq 1} \mathbb{Q}_n$ over \mathbb{Q} . Then $G_\infty \cong \Delta \times \Gamma$ where Δ is a finite group of order $p-1$ and $\Gamma \cong \mathbb{Z}_p$, the Galois group of $\mathbb{Q}_{p,\infty}$ over $\mathbb{Q}_p(\mu_p)$. We fix a topological generator γ of Γ and write χ for the cyclotomic character of G_∞ . Let $G_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $H_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p,\infty})$, where $\overline{\mathbb{Q}}_p$ denotes an algebraic closure of \mathbb{Q}_p .

Given a finite extension K of \mathbb{Q}_p with ring of integers \mathcal{O}_K , $\Lambda_{\mathcal{O}_K}(G_\infty)$ (respectively $\Lambda_{\mathcal{O}_K}(\Gamma)$) denotes the Iwasawa algebra of G_∞ (respectively Γ) over \mathcal{O}_K . We further write $\Lambda_K(G_\infty) = \Lambda_{\mathcal{O}_K}(G_\infty) \otimes \mathbb{Q}$ and $\Lambda_K(\Gamma) = \Lambda_{\mathcal{O}_K}(\Gamma) \otimes \mathbb{Q}$.

Given a module M over $\Lambda_{\mathcal{O}_K}(G_\infty)$ (respectively $\Lambda_K(G_\infty)$) and a character $\eta : \Delta \rightarrow \mathbb{Z}_p^\times$, M^η denotes the η -isotypical component of M . For any $m \in M$, we write m^η for the projection of m into M^η .

2. REPRESENTATIONS OF $G_{\mathbb{Q}_p}$

In this section we review some aspects of the theory of p -adic representations of $G_{\mathbb{Q}_p}$. Most of our account is reproduced from [Ber04] and [BLZ04, §2]. Let E be a finite extension of \mathbb{Q}_p with ring of integers \mathcal{O}_E . An E -linear representation of $G_{\mathbb{Q}_p}$ is a finite dimensional E -vector space V with a continuous E -linear action of $G_{\mathbb{Q}_p}$. We similarly have the notion of an \mathcal{O}_E -linear representation of $G_{\mathbb{Q}_p}$, which is a finitely-generated (not necessarily free) \mathcal{O}_E -module with a continuous \mathcal{O}_E -linear action of $G_{\mathbb{Q}_p}$. Define $\text{Rep}_E(G_{\mathbb{Q}_p})$ (respectively $\text{Rep}_{\mathcal{O}_E}(G_{\mathbb{Q}_p})$) to be the category of E -linear (respectively \mathcal{O}_E -linear) representations of $G_{\mathbb{Q}_p}$.

2.1. p -adic Hodge theory. In this section, we recall the definitions of some of Fountain's rings of periods. Let \mathbb{C}_p be the completion of $\overline{\mathbb{Q}}_p$ for the p -adic topology, endowed with the usual valuation v_p normalized such that $v_p(p) = 1$. Let

$$\tilde{\mathbb{E}} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, \dots) : (x^{(i+1)})^p = x^{(i)}\},$$

and let $\tilde{\mathbb{E}}^+$ be the set of $x \in \tilde{\mathbb{E}}$ such that $x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$. We can equip $\tilde{\mathbb{E}}$ naturally with the structure of an algebraically closed field of characteristic p : if $x = (x^{(i)})$ and $y = (y^{(i)})$, define $x + y$ and xy by

$$\begin{aligned} (x + y)^{(i)} &:= \lim_{j \rightarrow +\infty} (x^{(i+j)} + y^{(i+j)})^{p^j} \\ (xy)^{(i)} &:= x^{(i)}y^{(i)}. \end{aligned}$$

Define a complete valuation on $\tilde{\mathbb{E}}$ by $v_{\tilde{\mathbb{E}}}(x) = v_p(x^{(0)})$ if $x = (x^{(i)}) \in \tilde{\mathbb{E}}$. Let $\tilde{\mathbb{A}}^+ = W(\tilde{\mathbb{E}}^+)$ be the ring of Witt vectors of $\tilde{\mathbb{E}}^+$, and let $\tilde{\mathbb{B}}^+ = \tilde{\mathbb{A}}^+[p^{-1}]$. An element $x \in \tilde{\mathbb{B}}^+$ can then be written uniquely in the form

$$x = \sum_{i \gg -\infty} p^i [x_i],$$

where $x_i \in \tilde{\mathbb{E}}^+$ and $[x_i]$ denotes the Teichmüller lift. The ring $\tilde{\mathbb{B}}^+$ is equipped with the Witt vector Frobenius map φ (lifting the map $x \mapsto x^p$ on $\tilde{\mathbb{E}}^+$), and with a map

$$\theta : \tilde{\mathbb{B}}^+ \longrightarrow \mathbb{C}_p$$

via $\theta(\sum_{i \gg -\infty} p^i [x_i]) = \sum_{i \gg \infty} p^i x_i^{(0)}$. Fix an element $\varepsilon = (\varepsilon^{(n)}) \in \tilde{\mathbb{E}}^+$ with $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$. Let $\pi = [\varepsilon] - 1$, $\pi_1 = [\varphi^{-1}(\varepsilon)] - 1$ and $\omega = \frac{\pi}{\pi_1}$.

The ring \mathbb{B}_{dR}^+ is defined as $\mathbb{B}_{\text{dR}}^+ = \varprojlim \mathbb{B}^+ / \ker(\theta)^n$. It is a discrete valuation ring, and its maximal ideal is generated by $t = \log([\varepsilon])$. Define $\mathbb{B}_{\text{dR}} = \mathbb{B}_{\text{dR}}^+[t^{-1}]$ to be the fraction field of \mathbb{B}_{dR}^+ , which is equipped with an action of $G_{\mathbb{Q}_p}$ and a filtration defined by $\text{Fil}^i \mathbb{B}_{\text{dR}} = t^i \mathbb{B}_{\text{dR}}^+$.

Define the ring $\mathbb{B}_{\text{cris}}^+$ as

$$\mathbb{B}_{\text{cris}}^+ = \left\{ \sum_{n \geq 0} a_n \frac{\omega^n}{n!} \quad \text{where } a_n \in \tilde{\mathbb{B}}^+ \text{ is a sequence converging to 0} \right\},$$

and $\mathbb{B}_{\text{cris}} = \mathbb{B}_{\text{cris}}^+[t^{-1}]$. The ring \mathbb{B}_{cris} injects canonically into \mathbb{B}_{dR} , and it is endowed with the induced Galois action and filtration, as well with a continuous Frobenius φ which extends the map $\varphi : \tilde{\mathbb{B}}^+ \rightarrow \tilde{\mathbb{B}}^+$. If V is a \mathbb{Q}_p -linear representation of $G_{\mathbb{Q}_p}$, then $\mathbb{D}_{\text{cris}}(V) = (V \otimes \mathbb{B}_{\text{cris}})^{G_{\mathbb{Q}_p}}$ is a filtered φ -module of dimension $\leq \dim_{\mathbb{Q}_p}(V)$. We define V to be crystalline if equality holds.

If V is a \mathbb{Q}_p -linear representation of $G_{\mathbb{Q}_p}$, say that V is Hodge-Tate, with Hodge-Tate weights h_1, \dots, h_d , if we have a decomposition $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{i=1}^d \mathbb{C}_p(h_i)$. Say that V is positive if its Hodge-Tate weights are negative. It is easy to see that a crystalline representation V is Hodge-Tate, and that its Hodge-Tate weights are those integers h such that $\text{Fil}^{-h} \mathbb{D}_{\text{cris}}(V) \neq \text{Fil}^{1-h} \mathbb{D}_{\text{cris}}(V)$.

If V is an E -linear representation of $G_{\mathbb{Q}_p}$, then we define its Hodge-Tate weights to be the weights of the underlying \mathbb{Q}_p -vector space, and we say that V is crystalline if and only if the underlying \mathbb{Q}_p -linear representation is crystalline. In this case, $\mathbb{D}_{\text{cris}}(V)$ is an E -vector space, and the filtration and Frobenius are E -linear.

2.2. Crystalline representations and Wach modules. Let $\tilde{\mathbb{A}} = W(\tilde{\mathbb{E}})$, and let $\mathbb{A}_{\mathbb{Q}_p}$ be the completion of $\mathbb{Z}_p[[\pi]][\pi^{-1}]$ in $\tilde{\mathbb{A}}$ in the p -adic topology, so $\mathbb{A}_{\mathbb{Q}_p}$ is a complete discrete valuation ring with residue field $\mathbb{F}_p((\varepsilon - 1))$. Let \mathbb{B} be the completion of the maximal unramified extension of $\mathbb{B}_{\mathbb{Q}_p} = \mathbb{A}_{\mathbb{Q}_p}[p^{-1}]$ in $\tilde{\mathbb{B}}$, and define $\mathbb{A} = \mathbb{B} \cap \tilde{\mathbb{A}}$ and $\mathbb{B}^+ = \mathbb{B} \cap \tilde{\mathbb{B}}^+$. These rings are endowed with an action of $G_{\mathbb{Q}_p}$ and of the Frobenius operator φ . One can show that $(\mathbb{B}^+)^{H_{\mathbb{Q}_p}} = \mathbb{Z}_p[[\pi]][p^{-1}]$, which we denote by $\mathbb{B}_{\mathbb{Q}_p}^+$.

We define a left inverse $\psi : \mathbb{B} \rightarrow \mathbb{B}$ by $x \mapsto \varphi^{-1}(p^{-1} \text{Tr}_{\mathbb{B}/\varphi(\mathbb{B})}(x))$. If $x = f(\pi) \in \mathbb{B}_{\mathbb{Q}_p}$, then the value of $\psi(x)$ can also be calculated by

$$\varphi \circ \psi(x) = \frac{1}{p} \sum_{\zeta^p=1} f(\zeta(\pi+1) - 1).$$

Since the residual extension $\tilde{\mathbb{E}}/\varphi(\tilde{\mathbb{E}})$ is inseparable of degree p , ψ preserves \mathbb{A} and $\mathbb{A}_{\mathbb{Q}_p}$.

An étale (φ, G_∞) -module over $\mathbb{A}_{\mathbb{Q}_p}$ is a finitely generated $\mathbb{A}_{\mathbb{Q}_p}$ -module M , with semi-linear φ and a continuous action of G_∞ commuting with each other, such that $\varphi(M)$ generates M as an $\mathbb{A}_{\mathbb{Q}_p}$ -module. In [Fon90], Fontaine constructs a functor $T \rightarrow \mathbb{D}(T)$ which associates to every \mathbb{Z}_p -linear representation of $G_{\mathbb{Q}_p}$ an étale (φ, G_∞) -module over $\mathbb{A}_{\mathbb{Q}_p}$. Moreover, he shows that this functor is an equivalence of categories. By inverting p , one also gets an equivalence of categories between the category of \mathbb{Q}_p -linear p -adic representations and the category of étale (φ, G_∞) -modules over $\mathbb{B}_{\mathbb{Q}_p}$. The left inverse ψ of φ extends to the (φ, G_∞) -module.

If E is a finite extension of \mathbb{Q}_p , we extend the Frobenius and the action of G_∞ to $E \otimes \mathbb{B}_{\mathbb{Q}_p}$ by E -linearity. We then get an equivalence of categories from the category of E -linear (or \mathcal{O}_E -linear) representations to the category of étale (φ, G_∞) -modules over $E \otimes \mathbb{B}_{\mathbb{Q}_p}$ (resp. over $E \otimes \mathbb{A}_{\mathbb{Q}_p}$).

If V is a crystalline representation, we can say more about the (φ, G_∞) -module. Let $\mathbb{A}_{\mathbb{Q}_p}^+ = \mathbb{Z}_p[[\pi]]$ and $\mathbb{B}_{\mathbb{Q}_p}^+ = \mathbb{A}_{\mathbb{Q}_p}^+[p^{-1}]$ as above. The following result is shown in [Ber03, §II.1 and §III.4] and [BLZ04, §2]: If V is an E -linear representation, then V is crystalline with Hodge-Tate weights in $[a, b]$ if and only if there exists a (necessarily unique) $E \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathbb{Q}_p}^+$ -module $\mathbb{N}(V)$ contained in $\mathbb{D}(V)$ such that the following conditions are satisfied:

- (1) $\mathbb{N}(V)$ is free of rank $d = \dim_E(V)$ over $E \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathbb{Q}_p}^+$;
- (2) the action of G_∞ preserves $\mathbb{N}(V)$ and is trivial on $\mathbb{N}(V)/\pi\mathbb{N}(V)$;
- (3) $\varphi(\pi^b \mathbb{N}(V)) \subset \pi^b \mathbb{N}(V)$ and $\pi^b \mathbb{N}(V)/\varphi^*(\pi^b \mathbb{N}(V))$ is killed by q^{b-a} where $q = \frac{\varphi(\pi)}{\pi}$. (If M is a R -module equipped with a Frobenius φ where R is any ring, then $\varphi^*(M)$ denotes the R -module generated by $\varphi(M)$.)

If V is crystalline and positive, then we can take $b = 0$ above, so φ preserves $\mathbb{N}(V)$. In this case, if we endow $\mathbb{N}(V)$ with the filtration $\text{Fil}^i \mathbb{N}(V) = \{x \in \mathbb{N}(V) \mid \varphi(x) \in q^i \mathbb{N}(V)\}$, then $\mathbb{N}(V)/\pi\mathbb{N}(V)$ is a filtered E -linear φ -module, and as shown in [Ber03, §III.4] we have an isomorphism $\mathbb{N}(V)/\pi\mathbb{N}(V) \cong \mathbb{D}_{\text{cris}}(V)$.

If T is a $G_{\mathbb{Q}_p}$ -stable lattice in V , then $\mathbb{N}(T) = \mathbb{N}(V) \cap \mathbb{D}(T)$ is an $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+$ -lattice in $\mathbb{N}(V)$, and by [Ber03, §III.4] the functor $T \rightarrow \mathbb{N}(T)$ gives a bijection between the $G_{\mathbb{Q}_p}$ -stable lattices T in V and the $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+$ -lattices in $\mathbb{N}(V)$ satisfying

- (1) $\mathbb{N}(T)$ is free of rank $d = \dim_E(V)$ over $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+$;
- (2) the action of G_∞ preserves $\mathbb{N}(T)$;
- (3) $\varphi(\pi^b \mathbb{N}(T)) \subset \pi^b \mathbb{N}(T)$ and $\pi^b \mathbb{N}(T)/\varphi^*(\pi^b \mathbb{N}(T))$ is killed by q^{b-a} where $q = \frac{\varphi(\pi)}{\pi}$.

Let $\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+$ be the set of $f(\pi) \in \mathbb{Q}_p[[\pi]]$ such that $f(X)$ converges for all X in the open unit disc in \mathbb{C}_p . Note that $t \in \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+$. If V is a positive representation of $G_{\mathbb{Q}_p}$, then as shown in [Ber03, §I.5], we can recover $\mathbb{D}_{\text{cris}}(V)$ from $\mathbb{N}(V)$ as $\mathbb{D}_{\text{cris}}(V) = (\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ \otimes_{\mathbb{B}_{\mathbb{Q}_p}^+} \mathbb{N}(V))^{G_\infty}$. Moreover, the inclusion $\mathbb{D}_{\text{cris}}(V) \subset \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ \otimes_{\mathbb{B}_{\mathbb{Q}_p}^+} \mathbb{N}(V)$ gives rise to an isomorphism

$$\iota : \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ [t^{-1}] \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}}(V) \cong \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ [t^{-1}] \otimes_{\mathbb{B}_{\mathbb{Q}_p}^+} \mathbb{N}(V).$$

In [Ber02, proposition 2.12], Berger shows that for all $n \geq 0$ there is an injective map $\varphi^{-n}(\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+) \rightarrow \mathbb{B}_{\text{dR}}^+$, which is compatible with the natural map $\varphi^{-n}(\tilde{\mathbb{B}}^+) \rightarrow \mathbb{B}_{\text{dR}}^+$. It is characterized by the fact that it sends

π to $\varepsilon^{(n)} \exp(t/p^n) - 1$. Define a derivation $\partial : \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ \rightarrow \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+$ by $\partial = (1 + \pi) \frac{d}{d\pi}$. Under the map $\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ \rightarrow \mathbb{Q}_p[[t]]$ given by $\pi \mapsto \exp(t) - 1$, ∂ corresponds to the derivation $\frac{d}{dt}$.

If $z \in \mathbb{Q}_{p,n}((t)) \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}}(V)$, denote the constant coefficient of z by $\partial_V(z) \in \mathbb{Q}_{p,n} \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}}(V)$.

2.3. Iwasawa cohomology and the Fontaine isomorphism. If $T \in \text{Rep}_{\mathcal{O}_E}(G_{\mathbb{Q}_p})$, define

$$H_{\text{Iw}}^1(\mathbb{Q}_p, T) = \varprojlim_n H^1(\mathbb{Q}_{p,n}, T),$$

where the inverse limit is taken with respect to the corestriction maps. As shown by Fontaine (unpublished – for a reference see [CC99, Section II]), for any $T \in \text{Rep}_{\mathcal{O}_E}(G_{\mathbb{Q}_p})$, there is a canonical isomorphism of $\Lambda_{\mathcal{O}_E}(G_{\infty})$ -modules

$$(4) \quad h_{\mathbb{Q}_p, \text{Iw}}^1 : \mathbb{D}(T)^{\psi=1} \xrightarrow{\cong} H_{\text{Iw}}^1(\mathbb{Q}_p, T).$$

Similarly, for $V \in \text{Rep}_E(G_{\mathbb{Q}_p})$, define $H_{\text{Iw}}^1(\mathbb{Q}_p, V) = H_{\text{Iw}}^1(\mathbb{Q}_p, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where T is any $G_{\mathbb{Q}_p}$ -invariant lattice of V ; this is independent of the choice of T , and $h_{\mathbb{Q}_p, \text{Iw}}^1$ extends to an isomorphism of $\Lambda_E(G_{\infty})$ -modules $\mathbb{D}(V)^{\psi=1} \cong H_{\text{Iw}}^1(\mathbb{Q}_p, V)$.

3. THE COLEMAN MAPS

3.1. Positive crystalline representations. In this subsection, we shall define d Coleman maps for a d -dimensional positive crystalline representation V , depending on a choice of basis of the Wach module $\mathbb{N}(T)$ for a lattice T in V .

Let E be a finite extension of \mathbb{Q}_p . Let V be a positive crystalline d -dimensional E -linear representation of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights $-r_d \leq -r_{d-1} \leq \dots \leq -r_1 \leq 0$. We assume that V has no quotient isomorphic to $E(-r_d)$ and fix an \mathcal{O}_E -lattice T in V which is stable under $G_{\mathbb{Q}_p}$. Write $\mathbb{N}(T)$ for its Wach module, which is a free $\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+$ -module of rank d , whereas $\mathbb{N}(V) = \mathbb{N}(T) \otimes \mathbb{Q}_p$ is a free $E \otimes \mathbb{B}_{\mathbb{Q}_p}^+$ -module of rank d . Choose an $\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+$ -basis n_1, \dots, n_d of $\mathbb{N}(T)$ and write P for the matrix of φ with respect to this basis. Then

$$\begin{pmatrix} \varphi(n_1) \\ \vdots \\ \varphi(n_d) \end{pmatrix} = P^T \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix}$$

where A^T denotes the transpose of A if A is a square matrix. Moreover, by [BB10, section 3], the determinant of P is $q^{r_1 + \dots + r_d}$ up to a unit, where $q = \frac{\varphi(\pi)}{\pi}$ as above.

Let $m = \sum_{i=1}^d r_i$. Then, for $x \in \mathbb{D}(T(m))^{\psi=1}$, we have $x \in \mathbb{N}(T(m))^{\psi=1}$ by [Ber03, appendix A]. But $\mathbb{N}(T(m)) = \pi^{-m} \mathbb{N}(T) \otimes e_m$, where e_m is a vector space basis of $\mathbb{Z}_p(m)$. Hence, there exist unique $x_1, \dots, x_d \in \mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+$ such that

$$(5) \quad x = \pi^{-m} (x_1 \ \dots \ x_d) \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix} \otimes e_m.$$

Lemma 3.1. *For any $x \in \mathbb{D}(T(m))^{\psi=1}$, the entries of the row vector*

$$\mathbf{Col}(x) := (x_1 \ \dots \ x_d) q^m (P^T)^{-1} - (\varphi(x_1) \ \dots \ \varphi(x_d))$$

are elements of $(\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$.

Proof. Recall that the determinant of P is q^m up to a unit in $\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+$, so the entries of $\mathbf{Col}(x)$ are indeed elements of $\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+$. Since $\varphi(\pi) = \pi q$, (5) implies that

$$x = (x_1 \ \dots \ x_d) q^m (P^T)^{-1} \varphi(\pi^{-m}) \begin{pmatrix} \varphi(n_1) \\ \vdots \\ \varphi(n_d) \end{pmatrix} \otimes e_m.$$

Hence,

$$\psi(x) = \psi((x_1 \ \cdots \ x_d) q^m (P^T)^{-1}) \pi^{-m} \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix} \otimes e_m.$$

Therefore, $\psi(x) = x$ implies that

$$\psi((x_1 \ \cdots \ x_d) q^m (P^T)^{-1}) = (x_1 \ \cdots \ x_d).$$

Hence the result. \square

Definition 3.2. For $1 \leq i \leq d$, we define the i -th Coleman map $\text{Col}_i : \mathbb{D}(T(m))^{\psi=1} \rightarrow (\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$ by sending x to the i -th component of $\text{Col}(x)$.

Lemma 3.3. Let n_1, \dots, n_d and n'_1, \dots, n'_d be two bases of $\mathbb{N}(T)$ with $\begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix} = M'' \begin{pmatrix} n'_1 \\ \vdots \\ n'_d \end{pmatrix}$. Then, the

Coleman maps defined by these two bases, Col and Col' are related by $\text{Col}(x)\varphi(M'') = \text{Col}'(x)$ for all $x \in \mathbb{D}(T(m))^{\psi=1}$.

Proof. For any $x \in \mathbb{D}(T(m))^{\psi=1}$, write $x = x_1 n_1 + \cdots + x_d n_d = x'_1 n'_1 + \cdots + x'_d n'_d$. Then,

$$(x'_1 \ \cdots \ x'_d) = (x_1 \ \cdots \ x_d) M''$$

Let P and P' be the matrices of φ with respect to n_1, \dots, n_d and n'_1, \dots, n'_d respectively. Then $P^T M'' = \varphi(M'') P'^T$. Therefore,

$$\begin{aligned} \text{Col}'(x) &= (x'_1 \ \cdots \ x'_d) q^m (P'^T)^{-1} - (\varphi(x'_1) \ \cdots \ \varphi(x'_d)) \\ &= (x_1 \ \cdots \ x_d) q^m M'' (P'^T)^{-1} - (\varphi(x_1) \ \cdots \ \varphi(x_d)) \varphi(M'') \\ &= (x_1 \ \cdots \ x_d) q^m (P^T)^{-1} \varphi(M'') - (\varphi(x_1) \ \cdots \ \varphi(x_d)) \varphi(M''). \end{aligned}$$

Hence the lemma. \square

It is clear that we can extend Col_i to a map from $\mathbb{D}(V(m))^{\psi=1}$ to $(E \otimes \mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$. By an abuse of notation, we will write this map as Col_i as well. We now relate $\text{Col}(x)$ to $(1 - \varphi)(x)$. By writing down $\varphi(x)$, we have the following:

$$(6) \quad (1 - \varphi)(x) = \text{Col}(x) \cdot \varphi(\pi)^{-m} P^T \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix} \otimes e_m$$

Remark 3.4. We see from (6) that for any x as above, $(1 - \varphi)x \in (\varphi^* \mathbb{N}(T(m)))^{\psi=0}$.

Note that the maps Col_i are not $\Lambda(G_\infty)$ -homomorphisms under the canonical action of G_∞ on $(\mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$ because G_∞ acts non-trivially on the basis $\{n_i\}_{1 \leq i \leq d}$ of $\mathbb{N}(V)$. We deal with this problem using Theorem 3.5 below. Its proof is due to Laurent Berger; we quote it with his permission. In the case when V is the p -adic representation associated to a modular form with $v_p(a_p) \geq \lfloor \frac{k-2}{p-1} \rfloor$, we have independently found a proof of this result which uses the basis of $\mathbb{N}(V)$ constructed in [BLZ04]. It is more explicit than Berger's proof, and we give it in Section 4 since it will be needed to analyse the images of the Coleman maps. For notational simplicity, we take $E = \mathbb{Q}_p$ for the time being. Conceptually, there is no difficulty in extending the result to an E -linear representation.

Theorem 3.5. Let V be a crystalline p -adic representation of $G_{\mathbb{Q}_p}$ of dimension d , and let T be a $G_{\mathbb{Q}_p}$ -stable lattice in V . Then $(\varphi^* \mathbb{N}(T))^{\psi=0}$ is a free $\Lambda(G_\infty)$ -module of rank d . Moreover, if n_1^0, \dots, n_d^0 is a basis of $\mathbb{N}(T)$, then there exists a basis n_1, \dots, n_d such that $n_i \equiv n_i^0 \pmod{\pi}$ for all i and $(1 + \pi)\varphi(n_1), \dots, (1 + \pi)\varphi(n_d)$ forms a $\Lambda(G_\infty)$ -basis of $(\varphi^* \mathbb{N}(T))^{\psi=0}$.

Note that in this theorem we do not assume that V is positive. The proof of this result requires several preliminary lemmas. We assume without loss of generality that $\chi(\gamma) = 1 + p$. For $k \geq 0$, define

$$p_k = (1 - \gamma)(1 - \chi(\gamma)^{-1}\gamma) \dots (1 - \chi(\gamma)^{1-k}\gamma),$$

which is an element of $\Lambda(\Gamma)$.

Lemma 3.6. *If $a \in \mathbb{Z}_p$ and $x \in \mathbb{N}(T)$ and $f \in \mathbb{A}_{\mathbb{Q}_p}^+$ and $g \in G_\infty$, then*

$$(1 - ag)(fx) = ((1 - ag)f)x + ag(f)((1 - g)x).$$

Proof. Immediate. \square

Lemma 3.7. *The map $\mathfrak{M} : \Lambda(G_\infty) \rightarrow (\mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$ given by $f \mapsto f(1+\pi)$ is an isomorphism of $\Lambda(G_\infty)$ -modules, which takes $p_k \Lambda(G_\infty)$ to $\varphi(\pi)^k (\mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$.*

Proof. The first assertion is standard (we recall the relevant theory in section 5.1 below). Note that $\gamma(\pi) = \chi(\gamma)\pi + O(\pi^2)$, which implies that the image of $p_k \Lambda(G_\infty)$ is contained in $\varphi(\pi)^k (\mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$. Hence the surjection $\Lambda(G_\infty) \twoheadrightarrow (\mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$ gives a surjection $\Lambda(G_\infty)/p_k \twoheadrightarrow (\mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}/\varphi(\pi)^k$. Since both are free \mathbb{Z}_p -modules of rank $k(p-1)$, this must be an isomorphism. \square

Remark 3.8. *Following the terminology of [Ber03, §II.6], we refer to the inverse of \mathfrak{M} as the Mellin transform.*

Let n_1^0, \dots, n_d^0 be a basis of $\mathbb{N}(T)$. Since the action of G_∞ on $\mathbb{N}(T)$ is trivial modulo π , we have $(1 - g)n_i^0 \in \pi\mathbb{N}(T)$ for all $1 \leq i \leq d$ and for all $g \in G_\infty$.

Lemma 3.9. *Let g be a topological generator of G_∞ , and write $(1 - g)n_i^0 = \pi m_i$ for some $m_i \in \mathbb{N}(T)$. If we put $n_i = n_i^0 - \frac{\pi m_i}{1 - \chi(g)}$, then n_1, \dots, n_d is a basis of $\mathbb{N}(T)$, and $(1 - g)n_i \in \pi^2\mathbb{N}(T)$.*

Proof. Note that since $p \neq 2$ and g is a topological generator of G_∞ , $1 - \chi(g) \in \mathbb{Z}_p^\times$, so $n_i \in \mathbb{N}(T)$ for all i , and they are obviously a basis. Since $g(\pi) = \chi(g)\pi + O(\pi^2)$, this basis is designed such that $(1 - g)n_i \in \pi^2\mathbb{N}(T)$, and this implies that $(1 - g)n_i \in \pi^2\mathbb{N}(T)$. \square

Let \mathcal{N} be the $\Lambda(G_\infty)$ -submodule of $(\varphi^*(\mathbb{N}(T)))^{\psi=0}$ generated by $(1 + \pi)\varphi(n_1), \dots, (1 + \pi)\varphi(n_d)$.

Lemma 3.10. *Let $y \in (\varphi^*(\mathbb{N}(T)))^{\psi=0}$. Then there exist $\mathbf{n} \in \mathcal{N}$ and $z \in (\varphi^*(\mathbb{N}(T)))^{\psi=0}$ such that $y = \mathbf{n} + \varphi(\pi)z$.*

Proof. Write $y = \sum_{i=1}^d y_i \varphi(n_i)$ with $y_i \in (\mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$. By Lemma 3.7 we can write $y_i = b_i(1 + \pi)$ for some $b_i \in \Lambda(G_\infty)$, and Lemma 3.9 implies that $b_i n_i \equiv n_i \pmod{\pi^2\mathbb{N}(T)}$. Therefore, we have

$$\sum_{i=1}^d b_i((1 + \pi)\varphi(n_i)) - \sum_{i=1}^d y_i \varphi(n_i) \in \varphi(\pi)^2 (\varphi^*(\mathbb{N}(T)))^{\psi=0},$$

which is slightly better than the lemma. \square

Lemma 3.10 can be generalized to all $k \geq 0$:

Proposition 3.11. *Let $k \geq 0$ and $y \in \varphi(\pi)^k (\varphi^*(\mathbb{N}(T)))^{\psi=0}$. Then there exists $\mathbf{n} \in p_k \mathcal{N}$ and $z \in (\varphi^*(\mathbb{N}(T)))^{\psi=0}$ such that $y = \mathbf{n} + \varphi(\pi)^{k+1}z$.*

Proof. The case $k = 0$ is just Lemma 3.10. Assume that $k \geq 1$, and that the result is true for $k-1$. If $y = \sum_{i=1}^d y_i \varphi(n_i)$ with $y_i \in \varphi(\pi)^k (\mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$, then we can write $y_i = b_i(1 + \pi)$ with $b_i \in p_k \Lambda(G_\infty)$ by Lemma 3.7. By the definition of p_k , we can write $b_i = (1 - a\gamma)c_i$ with $a = \chi(\gamma)^{1-k}$ for some $c_i \in \Lambda(G_\infty)$. Moreover, $p_{k-1}|c_i$ for all i . Let $x_i = c_i(1 + \pi)$, then

$$\begin{aligned} \sum_{i=1}^d y_i \varphi(n_i) &= \sum_{i=1}^d ((1 - a\gamma)x_i) \varphi(n_i) \\ &= (1 - a\gamma) \left(\sum_{i=1}^d x_i \varphi(n_i) \right) - a \sum_{i=1}^d \gamma(x_i)((1 - \gamma)\varphi(n_i)) \end{aligned}$$

by Lemma 3.6. Let $z_0 := \sum_{i=1}^d \gamma(x_i)((1-\gamma)\varphi(n_i))$. By Lemma 3.9 and the fact that $p_{k-1}|c_i$ (so $\varphi(\pi)^{k-1}|x_i$), we have $z_0 \in \varphi(\pi)^{k+1}(\varphi^* \mathbb{N}(T))^{\psi=0}$.

Consider the element $\sum_{i=1}^d x_i \varphi(n_i)$ where $x_i = c_i(1+\pi)$ is divisible by $\varphi(\pi)^{k-1}$ by Lemma 3.7 as $p_{k-1}|c_i$. Therefore, by induction, we can write $\sum_{i=1}^d x_i \varphi(n_i)$ as $x + \varphi(\pi)^k w$ with $x \in p_{k-1} \mathcal{N}$ and $w \in (\varphi^* \mathbb{N}(T))^{\psi=0}$. If we set

$$\begin{aligned} \mathbf{n} &= (1-a\gamma)(x), \\ \varphi(\pi)^{k+1} z &= z_0 + (1-\chi(\gamma)^{-k}\gamma)(\varphi(\pi)^k w) \quad \text{and} \\ py_1 &= (\chi(\gamma)^{1-k} - \chi(\gamma)^{-k})\gamma(\varphi(\pi)^k w), \end{aligned}$$

then $y = \mathbf{n} + \varphi(\pi)^{k+1} z + py_1$ with $\mathbf{n} \in p_k \mathcal{N}$, $z \in (\varphi^* \mathbb{N}(T))^{\psi=0}$ and $y_1 \in \varphi(\pi)^k (\varphi^* \mathbb{N}(T))^{\psi=0}$.

Iterating this gives us $y_j \in (\varphi^* \mathbb{N}(T))^{\psi=0}$ and converging sequences $\mathbf{n}_j \in \mathcal{N}$ and $z_n \in (\varphi^* \mathbb{N}(T))^{\psi=0}$ such that

$$y = \mathbf{n}_j + \varphi(\pi)^{k+1} z_j + p^j y_j.$$

The proposition follows by taking \mathbf{n} and z to be the limits of \mathbf{n}_j and z_j , respectively. \square

Proof of Theorem 3.5. If $y \in (\varphi^* \mathbb{N}(T))^{\psi=0}$, the iterating Proposition 3.11 shows that for all $k \geq 0$ we can write

$$y = \mathbf{n}_0 + \mathbf{n}_1 + \cdots + \mathbf{n}_k + \varphi(\pi)^{k+1} z$$

with $\mathbf{n}_j \in p_j \mathcal{N}$. Passing to the limit over k shows that $y = \sum_{i \geq 0} \mathbf{n}_i \in \mathcal{N}$, which shows that $(1+\pi)\varphi(n_1), \dots, (1+\pi)\varphi(n_d)$ form a generating set of the $\Lambda(G_\infty)$ -module $(\varphi^* \mathbb{N}(T))^{\psi=0}$.

Finally, the map $\Lambda(G_\infty)^{\oplus d}/p_k \Lambda(G_\infty)^{\oplus d} \rightarrow (\varphi^* \mathbb{N}(T))^{\psi=0}/\varphi(\pi)^k (\varphi^* \mathbb{N}(T))^{\psi=0}$ is a surjective map between two \mathbb{Z}_p -modules of equal rank, so that it is injective, and therefore the kernel of $\Lambda(G_\infty)^{\oplus d} \rightarrow (\varphi^* \mathbb{N}(T))^{\psi=0}$ is equal to $\bigcap_{k \geq 0} p_k \Lambda(G_\infty)^d = 0$. This finishes the proof. \square

We now resume our assumption that V is a positive crystalline E -linear representation of $G_{\mathbb{Q}_p}$, with Hodge–Tate weights $-r_i$ such that $\sum_i r_i = m$, and $T \subset V$ an \mathcal{O}_E -lattice, as above. Applying theorem 3.5 to the representation $V(m)$, we find that for any basis n_1^0, \dots, n_d^0 of $\mathbb{N}(T)$, there is a basis n_1, \dots, n_d of $\mathbb{N}(T)$ with $n_i = n_i^0 \bmod \pi$ such that the vectors $(1+\pi)\varphi(\pi^{-m} n_i \otimes e_m)$ are a basis of $(\varphi^* \mathbb{N}(T(m)))^{\psi=0}$ as a $\Lambda_{\mathcal{O}_E}$ -module. With respect to such a basis n_1, \dots, n_d , we make the following definitions:

Definition 3.12. Define the Iwasawa transform to be the $\Lambda_{\mathcal{O}_E}(G_\infty)$ -equivariant isomorphism

$$\mathfrak{J} : (\varphi^* \mathbb{N}(T(m)))^{\psi=0} \longrightarrow \Lambda_{\mathcal{O}_E}(G_\infty)^{\oplus d}$$

determined by sending $(1+\pi)\varphi(n_i \otimes \pi^{-m} e_m)$ to $(0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is the i -th entry.

Definition 3.13. Define $\underline{\text{Col}} : \mathbb{N}(T(m))^{\psi=1} \rightarrow \Lambda_{\mathcal{O}_E}(G_\infty)^{\oplus d}$ as $\mathfrak{J} \circ (1-\varphi)$, and for $1 \leq i \leq d$, let $\underline{\text{Col}}_i : \mathbb{N}(T(m))^{\psi=1} \rightarrow \Lambda_{\mathcal{O}_E}(G_\infty)$ be the composition of $\underline{\text{Col}}$ with the projection onto the i -th component.

Note 3.14. For all $1 \leq i \leq d$, the map $\underline{\text{Col}}_i$ is $\Lambda_{\mathcal{O}_E}(G_\infty)$ -equivariant.

3.2. Comparison with \mathbb{D}_{cris} . We now give an alternative formula for the Coleman maps of the previous subsection using the comparison isomorphisms between the Wach module $\mathbb{N}(V)$ and $\mathbb{D}_{\text{cris}}(V)$.

Recall from section 2.2 that for any positive crystalline representation V we have a canonical isomorphism $\mathbb{N}(V)/\pi \mathbb{N}(V) \cong \mathbb{D}_{\text{cris}}(V)$ ([Ber03, § III.4]).

Lemma 3.15. Let V be a positive crystalline E -linear representation of $G_{\mathbb{Q}_p}$. Given any basis ν_1, \dots, ν_d of $\mathbb{D}_{\text{cris}}(V)$ over E , we can lift it to a basis of n_1, \dots, n_d of $\mathbb{N}(V)$ over $E \otimes \mathbb{B}_{\mathbb{Q}_p}^+$. Moreover, we may assume that $(1+\pi)\varphi(\pi^{-m} n_1 \otimes e_m), \dots, (1+\pi)\varphi(\pi^{-m} n_d \otimes e_m)$ is a $\Lambda_E(G_\infty)$ -basis of $(\varphi^* \mathbb{N}(V(m)))^{\psi=0}$.

Proof. Let T be a $G_{\mathbb{Q}_p}$ -stable \mathcal{O}_E -lattice in V . By theorem 3.5 above, we may choose a $\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+$ -basis $\bar{n}_1, \dots, \bar{n}_d$ of $\mathbb{N}(T)$ such that $(1+\pi)\varphi(\pi^{-m} \bar{n}_i \otimes e_m)$ is a $\Lambda_{\mathcal{O}_E}(G_\infty)$ -basis of $(\varphi^* \mathbb{N}(T(m)))^{\psi=0}$. Hence these elements are also a $\Lambda_E(G_\infty)$ -basis of $(\varphi^* \mathbb{N}(V(m)))^{\psi=0}$.

By the comparison isomorphism, the elements $\{\bar{\nu}_i := \bar{n}_i \bmod \pi : i = 1, \dots, d\}$ give a basis of $\mathbb{D}_{\text{cris}}(V)$ over E . Let $A \in GL_d(E)$ be the change of basis matrix from ν_1, \dots, ν_d to $\bar{\nu}_1, \dots, \bar{\nu}_d$. On applying A^{-1} to

$\bar{n}_1, \dots, \bar{n}_d$, we obtain a basis n_1, \dots, n_d of $\mathbb{N}(V)$ lifting ν_1, \dots, ν_d . Now it is clear that $(1 + \pi)\varphi(\pi^{-m}n_1 \otimes e_m), \dots, (1 + \pi)\varphi(\pi^{-m}n_d \otimes e_m)$ is a $\Lambda_E(G_\infty)$ -basis of $(\varphi^*\mathbb{N}(V(m)))^{\psi=0}$, since it differs from the original basis by the scalar matrix A^{-1} , which is clearly invertible in $\Lambda_E(G_\infty)$. \square

With respect to such a basis n_1, \dots, n_d of $\mathbb{N}(V)$, we can clearly define an Iwasawa transform and Coleman map as above but with E -coefficients,

$$\begin{aligned} \mathfrak{J} : (\varphi^*\mathbb{N}(V(m)))^{\psi=0} &\xrightarrow{\cong} \Lambda_E(G_\infty)^{\oplus d} \\ \underline{\text{Col}} : \mathbb{N}(V(m))^{\psi=1} &\longrightarrow \Lambda_E(G_\infty)^{\oplus d}, \end{aligned}$$

which are homomorphisms of $\Lambda_E(G_\infty)$ -modules.

Remark 3.16. *If T is an \mathcal{O}_E -lattice in V stable under $G_{\mathbb{Q}_p}$ and the \mathcal{O}_E -lattice in $\mathbb{D}_{\text{cris}}(V)$ spanned by ν_1, \dots, ν_d is the reduction of $\mathbb{N}(T)$, then we can define the Coleman maps integrally, as in the previous section. In section 4 below we will work with a specific basis ν_i for which such a lattice T can be explicitly constructed.*

Now, let ν_1, \dots, ν_d be a basis of $\mathbb{D}_{\text{cris}}(V)$ over E , and n_1, \dots, n_d a basis of $\mathbb{N}(V)$ lifting ν_1, \dots, ν_d as in lemma 3.15. We write A_φ for the matrix of φ on $\mathbb{D}_{\text{cris}}(V)$ with respect to the basis ν_1, \dots, ν_d . Again by [BB10, section 3], $E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+$ is a Bézout ring and

$$(7) \quad [(E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+) \otimes_{E \otimes \mathbb{B}_{\mathbb{Q}_p}^+} \mathbb{N}(V) : (E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+) \otimes_E \mathbb{D}_{\text{cris}}(V)] = \left[\left(\frac{t}{\pi} \right)^{r_1}; \dots; \left(\frac{t}{\pi} \right)^{r_d} \right].$$

In other words, there exists $E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+$ -bases w_1, \dots, w_d and v_1, \dots, v_d for $(E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+) \otimes_{E \otimes \mathbb{B}_{\mathbb{Q}_p}^+} \mathbb{N}(V)$ and $(E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+) \otimes_E \mathbb{D}_{\text{cris}}(V)$ respectively such that $v_i = (t/\pi)^{r_i} w_i$ for $i = 1, \dots, d$. Therefore, the change of basis matrix $M' \in M_d(E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)$ with

$$(8) \quad \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} = M' \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix},$$

has determinant $(t/\pi)^m$ up to a unit in $E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+$. Moreover, since n_1, \dots, n_d lifts ν_1, \dots, ν_d , we have $M'|_{\pi=0} = I$, the identity matrix. The compatibility of the action of φ implies that

$$(9) \quad \varphi(M')P^T = A_\varphi^T M',$$

where P is the matrix of φ on $\mathbb{N}(V)$ with respect to the basis n_1, \dots, n_d as in the previous subsection. We can now rewrite (5):

$$(10) \quad x = (x_1 \quad \cdots \quad x_d) \cdot \left(\frac{t}{\pi} \right)^m M'^{-1} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} \otimes t^{-m} e_m$$

with $(t/\pi)^m M'^{-1} \in M_d(E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)$ and $\nu_i \otimes t^{-m} e_m, i = 1, \dots, d$ a basis of $\mathbb{D}_{\text{cris}}(V(m))$.

Rewriting (6) using this, we see that

$$(11) \quad (1 - \varphi)(x) = \underline{\text{Col}}(x) \cdot \left(\frac{t}{\pi q} \right)^m P^T M'^{-1} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} \otimes t^{-m} e_m.$$

3.3. Supersingular modular forms. We now apply the theory of Coleman maps developed above to the Galois representations attached to modular forms.

Let $f = \sum a_n q^n$ be a normalized new eigenform of weight k and character ϵ . Let p be an odd prime which does not divide the level of f . For simplicity, we will always assume that $\epsilon(p) = 1$. In particular $a_p = \bar{a}_p$. We write $E = \mathbb{Q}_p(a_n : n \geq 1)$, which is the completion of the coefficient field F of f at the prime above p determined our choice of embeddings. Then, by Deligne [Del69], we can associate to f a 2-dimensional E -linear representation V_f of $G_{\mathbb{Q}}$. Moreover, when restricted to $G_{\mathbb{Q}_p}$, V_f is crystalline and its de Rham filtration is given by

$$(12) \quad \mathbb{D}_{\text{cris}}^i(V_f) = \begin{cases} E\nu_1 \oplus E\nu_2 & \text{if } i \leq 0 \\ E\nu_1 & \text{if } 1 \leq i \leq k-1 \\ 0 & \text{if } i \geq k \end{cases}$$

for some basis ν_1, ν_2 over E . We further assume that $v_p(a_p) \neq 0$, i.e. f is supersingular at p . Then ν_1 is not an eigenvector of φ by [Kat04, Theorem 16.6] and we may choose $\nu_2 = p^{1-k}\varphi(\nu_1)$ so that the matrix A_φ of φ with respect to the basis ν_1, ν_2 is given by

$$\begin{pmatrix} 0 & -1 \\ p^{k-1} & a_p \end{pmatrix}$$

since $\varphi^2 - a_p\varphi + p^{k-1} = 0$ (c.f. [Sch90]). We call such a basis a ‘good basis’ for $\mathbb{D}_{\text{cris}}(V_f)$.

Let $\bar{\nu}_1$ and $\bar{\nu}_2$ be a ‘good basis’ of $\mathbb{D}_{\text{cris}}(V_{\bar{f}})$. Then, the matrix of φ with respect to this basis is equal to A_φ also since $a_p = \bar{a}_p$.

Note that $V_{\bar{f}}$ has Hodge-Tate weights 0 and $-k+1$, so it is positive. Fix a basis n_1, n_2 of $\mathbb{N}(V_{\bar{f}})$ satisfying the conditions in Lemma 3.15, so $\begin{pmatrix} \bar{\nu}_1 \\ \bar{\nu}_2 \end{pmatrix} = M' \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ with $M'|_{\pi=0} = I$. We obtain two pairs of Coleman maps associated to f :

$$\begin{aligned} \text{Col}_i : \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1} &\longrightarrow (E \otimes \mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}, \\ \underline{\text{Col}}_i : \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1} &\longrightarrow \Lambda_E(G_\infty), \end{aligned}$$

for $i = 1, 2$.

Recall the isomorphism (4) above:

$$h_{\mathbb{Q}_p, \text{Iw}}^1 : \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1} \cong H_{\text{Iw}}^1(\mathbb{Q}_p, V_{\bar{f}}(k-1)).$$

We can therefore consider the localization of Kato’s zeta element \mathbf{z}^{Kato} from [Kat04] (see section 6.1 below), which *a priori* is an element of $H_{\text{Iw}}^1(\mathbb{Q}_p, V_{\bar{f}}(k-1))$, as an element of $\mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$. We can now define two pairs of p -adic L -functions:

Definition 3.17. For $i = 1, 2$, define $L_{p,i} = \text{Col}_i(\mathbf{z}^{\text{Kato}}) \in (E \otimes \mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$ and $\tilde{L}_{p,i} = \underline{\text{Col}}_i(\mathbf{z}^{\text{Kato}}) \in \Lambda_E(G_\infty)$ where \mathbf{z}^{Kato} is the localization of the Kato zeta element.

The reason why we consider $V_{\bar{f}}$ instead of V_f will become apparent in section 3.4 below. In addition, below is a list of assumptions which we will need later when we prove different results.

- **Assumption (A):** $k \geq 3$ or $a_p = 0$.
- **Assumption (B):** a_p is not of the form $p^j + p^{k-2-j}$ for some integer $1 \leq j \leq k-3$.
- **Assumption (C):** $v_p(a_p) > \lfloor (k-2)/(p-1) \rfloor$.
- **Assumption (D):** $p \geq k-1$.

3.4. Relation to the Perrin-Riou pairing. Let α and β be the roots of the quadratic $X^2 - a_p X + p^{k-1}$. By the work of Amice–Vélu and Vishik cited in the introduction, we can associate to α and β p -adic L -functions $L_{p,\alpha}$ and $L_{p,\beta}$ respectively; see [MTT86, §11] for an account of the construction. We will relate them to $L_{p,i}$, $i = 1, 2$, as defined above. We first prove some preliminary results on general crystalline representations.

Let γ be a topological generator of Γ . Define

$$\mathcal{H}(G_\infty) = \{f(\gamma - 1) \mid f(X) \in \mathbb{Q}_p[\Delta][[X]] \text{ such that } f \text{ converges for all } X \in \mathbb{C}_p \text{ with } |X| < 1\}.$$

We can identify $\mathcal{H}(G_\infty)$ with $(\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0}$ via the map

$$(13) \quad \begin{aligned} \mathfrak{M} : \mathcal{H}(G_\infty) &\longrightarrow (\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0} \\ f(\gamma - 1) &\longmapsto f(\gamma - 1)(\pi + 1), \end{aligned}$$

where any $g \in G_\infty$ acts on π by $(\pi + 1)^{\chi(g)} - 1$. As shown in [PR01, B.2.8], this map is a bijection, extending the isomorphism $\Lambda(G_\infty) \rightarrow (\mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$ of Lemma 3.7. For $r \geq 1$, define

$$\mathcal{H}_r^{\text{temp}} = \left\{ \sum_{\sigma \in \Delta} \sum_{n \geq 0} c_{n, \sigma} \sigma X^n : \lim_{n \rightarrow +\infty} \frac{|c_{n, \sigma}|_p}{n^r} = 0 \right\}.$$

Let $\mathcal{H}^{\text{temp}} = \bigcup_{r \geq 1} \mathcal{H}_r^{\text{temp}}$, and define $\mathcal{H}^{\text{temp}}(G_\infty) = \{f(\gamma - 1) \mid f(X) \in \mathcal{H}^{\text{temp}}\}$.

Let V be any crystalline E -linear representation of $G_{\mathbb{Q}_p}$, and let h be a positive integer such that $\text{Fil}^{-h} \mathbb{D}_{\text{cris}}(V) = \mathbb{D}_{\text{cris}}(V)$. Denote by

$$\Omega_{V, h} : (\mathcal{H}^{\text{temp}}(G_\infty) \otimes \mathbb{D}_{\text{cris}}(V))^{\Sigma=0} \longrightarrow \mathcal{H}^{\text{temp}}(G_\infty) \otimes_{\Lambda_{\mathbb{Q}_p}} H_{\text{Iw}}^1(\mathbb{Q}_p, V)$$

Perrin-Riou's exponential map as constructed in [PR94]. Here,

$$\Sigma : \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}}(V) \rightarrow \bigoplus_{k=0}^h (\mathbb{D}_{\text{cris}}(V)/(1 - p^k \varphi))(k)$$

is the map sending $f \in \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}}(V)$ to the class of $\oplus \partial^k(f)(0)$, where $\partial = (1 + \pi) \frac{d}{d\pi}$ is the derivation on $\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+$ defined in §2.2. Since $\Omega_{V, h}$ is a homomorphism of $\mathcal{H}^{\text{temp}}(G_\infty)$ -modules, we can extend scalars to get

$$(14) \quad \Omega_{V, h} : ((\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0} \otimes \mathbb{D}_{\text{cris}}(V))^{\Sigma=0} \longrightarrow \mathcal{H}(G_\infty) \otimes_{\Lambda_{\mathbb{Q}_p}} H_{\text{Iw}}^1(\mathbb{Q}_p, V),$$

where we identify $(\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0}$ with $\mathcal{H}(G_\infty)$ via \mathfrak{M} .

Remark 3.18. We will only apply (14) to elements in which lie in the image of $\mathcal{H}^{\text{temp}}(G_\infty) \otimes \mathbb{D}_{\text{cris}}(V)$ under \mathfrak{M} , so we can refer to [PR94] for the properties of $\Omega_{V, h}$. The reason for extending scalars to $\mathcal{H}(G_\infty)$ is that we want to be able to use Berger's description of the exponential map in [Ber03, §II.5].

Recall that we have chosen a p -power compatible system $\varepsilon^{(n)}$, $n \geq 0$, of p -power roots of unity.

Proposition 3.19. Assume that V is a crystalline representation of $G_{\mathbb{Q}_p}$. Let $h \geq 1$ such that $\mathbb{D}_{\text{cris}}^{-h}(V) = \mathbb{D}_{\text{cris}}(V)$ and p^{-j} is not an eigenvalue of φ on $\mathbb{D}_{\text{cris}}(V)$ for $j \in \mathbb{Z}$ with $0 \leq j \leq h$. Then, for all $v \in \mathbb{D}_{\text{cris}}(V)$, the projection to the n -th local cohomology $H^1(\mathbb{Q}_{p, n}, V)$ of $\frac{1}{(h-1)!} \Omega_{V, h}((1 + \pi) \otimes v)$ is given by

$$(15) \quad \begin{cases} p^{-n} \exp_{F_n, V} \left(\sum_{m=0}^{n-1} \varepsilon^{(n-m)} \otimes \varphi^{m-n}(v) + (1 - \varphi)^{-1}(v) \right) & \text{if } n \geq 1 \\ \exp_{\mathbb{Q}_p, V} \left(\left(1 - \frac{\varphi^{-1}}{p} \right) (1 - \varphi)^{-1}(v) \right) & \text{if } n = 0. \end{cases}$$

Proof. Let $g \in (\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0} \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}}(V)$. We write $\Delta_j(g) = \partial^j(g)(0)$ and

$$\tilde{g} = g - \sum_{j=0}^h \frac{1}{j!} \log_p^j \Delta_j(g).$$

By [PR94, section 2.2], the sum $\sum_{n=0}^{\infty} \varphi^n(\tilde{g})$ converges. A solution to $(1 - \varphi)G = g$ with $G \in (\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}})^{\psi=1}$ is given by

$$G = \sum_{n=0}^{\infty} \varphi^n(\tilde{g}) + \sum_{j=0}^h \frac{1}{j!} \log_p^j v_j$$

where $v_j \in \mathbb{D}_{\text{cris}}(V)$ is such that $\Delta_j(g) = (1 - p^j \varphi)v_j$. Now, take $g = (1 + \pi) \otimes v$, so $\Delta_j(g) = v$ for all j . Let n be a positive integer, then

$$(16) \quad \varphi^m(\tilde{g})(\varepsilon^{(n)} - 1) = \begin{cases} (\varepsilon^{(n-m)} - 1) \otimes \varphi^m(v) & \text{if } m < n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned} G(\varepsilon^{(n)} - 1) &= \sum_{m=0}^{n-1} (\varepsilon^{(n-m)} - 1) \otimes \varphi^m(v) + (1 - \varphi)^{-1}(v) \\ &= \sum_{m=0}^{n-1} \varepsilon^{(n-m)} \otimes \varphi^m(v) + (1 - \varphi)^{-1} \varphi^n(v) \end{aligned}$$

Hence, by the main result in [PR94], the n -th component of $\frac{1}{(h-1)!} \Omega_{V,h}((1 + \pi) \otimes v)$ is given by the image of

$$(17) \quad p^{-n} \varphi^{-n} G(\varepsilon^{(n)} - 1) = \frac{1}{p^n} \left(\sum_{m=0}^{n-1} \varepsilon^{(n-m)} \otimes \varphi^{m-n}(v) + (1 - \varphi)^{-1}(v) \right)$$

under the map $\exp_{\mathbb{Q}_{p,n}, V}$. For the 0-th level, it is given by the image of

$$\begin{aligned} \text{Tr}_{\mathbb{Q}_{p,1}/\mathbb{Q}_p} \left(\frac{1}{p} \varphi^{-1} G(\varepsilon^{(1)} - 1) \right) &= \frac{1}{p} \text{Tr}_{\mathbb{Q}_{p,1}/\mathbb{Q}_p} (\varepsilon^{(1)} \otimes \varphi^{-1}(v) + (1 - \varphi)^{-1}(v)) \\ &= \frac{1}{p} (-1 \otimes \varphi^{-1}(v) + (p-1)(1 - \varphi)^{-1}(v)) \\ &= \left(1 - \frac{\varphi^{-1}}{p} \right) (1 - \varphi)^{-1}(v). \end{aligned}$$

under the map $\exp_{\mathbb{Q}_p, V}$, so we are done. \square

Define the Perrin-Riou pairing $\langle \cdot, \cdot \rangle_V$ by

$$\begin{aligned} \langle \cdot, \cdot \rangle_V : H_{\text{Iw}}^1(\mathbb{Q}_p, V) \times H_{\text{Iw}}^1(\mathbb{Q}_p, V^*(1)) &\longrightarrow \Lambda_E(G_\infty), \\ \langle (x_n), (y_n) \rangle_V &= \varprojlim \Sigma_{\tau \in G_{\mathbb{Q}_p}/G_{\mathbb{Q}_p}^{p^n}} (\tau(x_n) \cup y_n) \tau. \end{aligned}$$

Remark 3.20. In [PR94], the pairing is defined by

$$\langle (x_n), (y_n) \rangle_V = \varprojlim \Sigma_{\tau \in G_{\mathbb{Q}_p}/G_{\mathbb{Q}_p}^{p^n}} (\tau^{-1}(x_n) \cup y_n) \tau.$$

We use the different convention so that the map $\mathcal{L}_{h,z}$ defined in (18) below is a $\Lambda(G_\infty)$ -homomorphism.

We can extend the pairing $\langle \cdot, \cdot \rangle_V$ to

$$\langle \cdot, \cdot \rangle_V : \left(\mathcal{H}(G_\infty) \otimes_{\Lambda_{\mathbb{Q}_p}} H_{\text{Iw}}^1(\mathbb{Q}_p, V) \right) \times \left(\mathcal{H}(G_\infty) \otimes_{\Lambda_{\mathbb{Q}_p}} H_{\text{Iw}}^1(\mathbb{Q}_p, V^*(1)) \right) \longrightarrow \mathcal{H}(G_\infty).$$

Any $z \in ((\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0} \otimes \mathbb{D}_{\text{cris}}(V))^{\Sigma=0}$ therefore defines a map

$$(18) \quad \begin{aligned} \mathcal{L}_{h,z} : H_{\text{Iw}}^1(\mathbb{Q}_p, V^*(1)) &\longrightarrow \mathcal{H}(G_\infty), \\ (y_n)_{n \geq 0} &\longmapsto \langle \Omega_{h,V}(z), (y_n) \rangle_V. \end{aligned}$$

As recalled in section 2.3 above, for any p -adic representation V of $G_{\mathbb{Q}_p}$ we have a canonical isomorphism

$$h_{\mathbb{Q}_p, \text{Iw}}^1 : \mathbb{D}(V)^{\psi=1} \cong H_{\text{Iw}}^1(\mathbb{Q}_p, V).$$

Lemma 3.21. For all $j \in \mathbb{Z}$ and for all $y \in \mathbb{D}(V)^{\psi=1}$ and $y' \in \mathbb{D}(V^*(1))^{\psi=1}$, we have

$$\partial^j \langle h_{\mathbb{Q}_p, \text{Iw}}^1(y), h_{\mathbb{Q}_p, \text{Iw}}^1(y') \rangle_V = \langle h_{\mathbb{Q}_p, \text{Iw}}^1(y \otimes e_j), h_{\mathbb{Q}_p, \text{Iw}}^1(y' \otimes e_{-j}) \rangle_{V(j)}.$$

Proof. See Lemme ii) Section 3.6 in [PR94]. \square

We now return to the setting in Section 3.3. We will apply Perrin-Riou's theory that we recalled above to the crystalline representation $V_f(1)$. In particular, $V_f(1)^*(1) \cong V_{\bar{f}}(k-1)$. By (12), we can take $h = 1$. Note that φ acts on $\mathbb{D}_{\text{cris}}(V_f(1))$ by $\begin{pmatrix} 0 & -p^{-1} \\ p^{k-2} & p^{-1}a_p \end{pmatrix}$ with respect to a 'good basis' $\nu_i \otimes t^{-1}e_1$, $i = 1, 2$ as chosen in Section 3.3. But $a_p \neq p + p^{k-2}$ by the Weil bound, so both $1 - \varphi$ and $1 - p\varphi$ are isomorphisms on $\mathbb{D}_{\text{cris}}(V_f(1))$ and $\Sigma = 0$. Let $\bar{\nu}_1, \bar{\nu}_2$ be a 'good basis' for $\mathbb{D}_{\text{cris}}(V_{\bar{f}})$. We can identify $\mathbb{D}_{\text{cris}}(V_f)$ with $\mathbb{D}_{\text{cris}}(V_f(1))$ (resp. $\mathbb{D}_{\text{cris}}(V_{\bar{f}})$ with $\mathbb{D}_{\text{cris}}(V_{\bar{f}}(k-1))$) via $\nu_i \mapsto \nu_i \otimes e_1 t^{-1}$ (resp. $\bar{\nu}_i \mapsto \bar{\nu}_i \otimes e_{k-1} t^{1-k}$). Under these identifications, the natural pairing

$$(19) \quad [\ , \] : \mathbb{D}_{\text{cris}}(V_f(1)) \times \mathbb{D}_{\text{cris}}(V_{\bar{f}}(k-1)) \rightarrow \mathbb{D}_{\text{cris}}(E(1)) = E \cdot e_1 t^{-1}$$

satisfies $[\nu_1, \bar{\nu}_1] = 0$. By applying φ , we have $[\nu_2, \bar{\nu}_2] = 0$, too. We also have $[\nu_1, \bar{\nu}_2] = -[\nu_2, \bar{\nu}_1] \neq 0$. Without loss of generality, we may assume that $[\nu_1, \bar{\nu}_2] = -[\nu_2, \bar{\nu}_1] = 1$

Let $x \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$. It follows from the construction of the Coleman maps in Section 3.3 that if we let

$$(20) \quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \left(\frac{t}{\pi q} \right)^{k-1} P^T M'^{-1},$$

then, by (11), $(1 - \varphi)(x)$ can be written as

$$(21) \quad (1 - \varphi)(x) = (y_1 m_{11} + y_2 m_{21}) \bar{\nu}_1 \otimes t^{1-k} e_{k-1} + (y_1 m_{12} + y_2 m_{22}) \bar{\nu}_2 \otimes t^{1-k} e_{k-1},$$

where $y_i = \text{Col}_i(x)$ for $i = 1, 2$.

Proposition 3.22. *On identifying with their images under \mathfrak{M} , we have*

$$(22) \quad \langle \Omega_{V_f(1),1}((1 + \pi) \otimes \nu_1), x \rangle_{V_f(1)} = y_1 m_{12} + y_2 m_{22},$$

$$(23) \quad -\langle \Omega_{V_f(1),1}((1 + \pi) \otimes \nu_2), x \rangle_{V_f(1)} = y_1 m_{11} + y_2 m_{21}.$$

The rest of this section is devoted to proving this result. We follow closely Berger's proof of Perrin-Riou's explicit reciprocity law in [Ber03]. We first make the following definition: let $V \in \text{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$. For an element $x \in H_{\text{Iw}}^1(\mathbb{Q}_p, V)$, define $h_{\mathbb{Q}_p, V}^1(x)$ to be the image of x under the projection map $H_{\text{Iw}}^1(\mathbb{Q}_p, V) \rightarrow H^1(\mathbb{Q}_p, V)$.

Recall also the map ∂_V defined in subsection 2.2: for $z \in \mathbb{Q}_{p,n}((t)) \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}}(V_f(1+j))$, we denote the constant coefficient of z by $\partial_{V_f(1+j)}(z) \in \mathbb{Q}_{p,n} \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}}(V_f(1+j))$.

Lemma 3.23. *Let $i \in \{1, 2\}$, and choose $\mathfrak{y}_i \in (\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}}(V_f(1)))^{\psi=1}$ such that $(1 - \varphi)\mathfrak{y}_i = (1 + \pi) \otimes \nu_i$. Then*

$$\begin{aligned} h_{\mathbb{Q}_p, V_f(1+j)}^1 \Omega_{V_f(1+j), 1+j} (\partial^{-j} (1 + \pi) \otimes \nu_i \otimes t^{-j} e_j) = \\ j! \exp_{\mathbb{Q}_p, V_f(1+j)} \left(\left(1 - \frac{\varphi^{-1}}{p}\right) \partial_{V_f(1+j)} (\partial^{-j} \mathfrak{y}_i \otimes t^{-j} e_j) \right). \end{aligned}$$

Proof. By Proposition 3.19, we need to prove that $\partial_{V_f(1+j)} (\partial^{-j} \mathfrak{y}_i \otimes t^{-j} e_j) = (1 - \varphi)^{-1} (\nu_i \otimes t^{-j} e_j)$. Note that φ commutes with $\partial_{V_f(1+j)}$ and $\varphi \circ \partial^{-j} = p^j \partial^{-j} \circ \varphi$, so

$$(1 - \varphi) \partial_{V_f(1+j)} (\partial^{-j} \mathfrak{y}_i \otimes t^{-j} e_j) = \partial_{V_f(1+j)} (\partial^{-j} (1 + \pi) \otimes \nu_i \otimes t^{-j} e_j).$$

Note that $\partial(1 + \pi) = 1 + \pi$, so $\partial_{V_f(1+j)} (\partial^{-j} (1 + \pi) \otimes \nu_i \otimes t^{-j} e_j) = \nu_i \otimes t^{-j} e_j$. Also, as observed above, $1 - \varphi$ is invertible on $\mathbb{D}_{\text{cris}}(V_f(1))$, which proves the result. \square

We can now prove Proposition 3.22. We will only prove (23) here; the proof of (22) is analogous.

Proof. For $i = 1, 2$, let $\mathfrak{y}_i \in (\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}}(V_f(1)))^{\psi=1}$ such that $(1 - \varphi)\mathfrak{y}_i = (1 + \pi) \otimes \nu_i$. By p -adic interpolation it is sufficient to show that

$$\partial^j (\langle \Omega_{V_f(1),1}((1 + \pi) \otimes \nu_2), h_{\mathbb{Q}_p, \text{Iw}}^1(x) \rangle_{V_f(1)}) (0) = \partial^j (y_1 m_{12} + y_2 m_{22}) (0)$$

for all $j \gg 0$. We have

$$\begin{aligned} \partial^j (\langle \Omega_{V_f(1),1}((1+\pi) \otimes \nu_2), x \rangle_{V_f(1)}) &= \langle \Omega_{V_f(1),1}((1+\pi) \otimes \nu_2) \otimes e_j, h_{\text{Iw},V_f(k-1-j)}^1(x \otimes e_{-j}) \rangle_{V_f(1+j)} \\ (24) \quad &= (-1)^j \langle \Omega_{V_f(1+j),1+j}(\partial^{-j}(1+\pi) \otimes \nu_2 \otimes t^{-j}e_j), h_{\text{Iw},V_f(k-1-j)}^1(x \otimes e_{-j}) \rangle_{V_f(1+j)} \end{aligned}$$

by Lemma 3.21 and the properties of Ω (c.f. p. 119, Théorème (B)(ii) in [PR94]). Hence

$$\begin{aligned} \partial^j (\langle \Omega_{V_f(1),1}((1+\pi) \otimes \nu_2), x \rangle_{V_f(1)}) (0) \\ (25) \quad &= (-1)^j \langle h_{\mathbb{Q}_p,V_f(1+j)}^1 \Omega_{V_f(1+j),1+j}(\partial^{-j}(1+\pi) \otimes \nu_2 \otimes t^{-j}e_j), h_{\mathbb{Q}_p,V_f(k-1-j)}^1(x \otimes e_{-j}) \rangle_{V_f(1+j)} \\ (26) \quad &= j! \left\langle \exp_{\mathbb{Q}_p,V_f(1+j)} \left(\left(1 - \frac{\varphi^{-1}}{p}\right) \partial_{V_f(1+j)}(\partial^{-j}\eta_2 \otimes t^{-j}e_j) \right), h_{\mathbb{Q}_p,V_f(k-1-j)}^1(x \otimes e_{-j}) \right\rangle_{V_f(1+j)} \\ (27) \quad &= j! \left[\left(1 - \frac{\varphi^{-1}}{p}\right) \partial_{V_f(1+j)}(\partial^{-j}\eta_2 \otimes t^{-j}e_j), \exp_{\mathbb{Q}_p,V^*(1+j)}^* h_{\mathbb{Q}_p,V_f(k-1-j)}^1(x \otimes e_{-j}) \right]_{V_f(1+j)} \\ (28) \quad &= j! \left[\left(1 - \frac{\varphi^{-1}}{p}\right) \partial_{V_f(1+j)}(\partial^{-j}\eta_2 \otimes t^{-j}e_j), \left(1 - \frac{\varphi^{-1}}{p}\right) \partial_{V_f(k-1-j)}(x \otimes e_{-j}) \right]_{V_f(1+j)} \\ (29) \quad &= j! [\partial_{V_f(1+j)}(\partial^{-j}(1+\pi) \otimes \nu_2 \otimes t^{-j}e_j), \partial_{V_f(k-1-j)}((1-\varphi)x \otimes e_{-j})]_{V_f(1+j)} \end{aligned}$$

The equalities can be explained as follows:

- the first equality is immediate from (24) and the construction of $\langle \cdot, \cdot \rangle_V$;
- the implication (25) \Rightarrow (26) follows from Lemma 3.23;
- the implication (26) \Rightarrow (27) is the duality between $\exp_{F,V_f(1+j)}$ and $\exp_{F,V_f(1+j)}^*$;
- the implication (27) \Rightarrow (28) follows from [Ber03, Theorem II.6], and
- (29) follows from (28) since $1-\varphi$ is the adjoint of $1 - \frac{\varphi^{-1}}{p}$ under the pairing $[\cdot, \cdot]$.

Now $\partial(1+\pi) = 1+\pi$, which implies that $\partial^{-j}(1+\pi) \otimes \nu_2 \otimes t^{-j}e_j = (1+\pi) \otimes \nu_2 \otimes t^{-j}e_j$ and hence

$$\partial_{V_f(1+j)}(\partial^{-j}(1+\pi) \otimes \nu_2 \otimes t^{-j}e_j) = \nu_2 \otimes t^{-j}e_j.$$

By (21), we can write

$$(1-\varphi)x = (y_1 m_{11} + y_2 m_{21})\bar{\nu}_1 + (y_1 m_{12} + y_2 m_{22})\bar{\nu}_2.$$

Recall that by construction, we have $[\nu_2, \bar{\nu}_1] = -1$ and $[\nu_i, \bar{\nu}_i] = 0$ for $i = 1, 2$. It follows that if we write $-(y_1 m_{11} + y_2 m_{21}) = \sum_{i \geq 0} c_i t^i$ with $c_i \in \mathbb{Q}_p$, then (29) is equal to $j! c_j$. Since also $-\partial^j(y_1 m_{11} + y_2 m_{21})(0) = j! c_j$, this finishes the proof of (23). \square

We can summarize the results of this section by the following corollary:

Corollary 3.24. *We have a commutative diagram*

$$\begin{array}{ccc} \mathbb{N}(V)^{\psi=1} & \xrightarrow{h_{\mathbb{Q}_p,\text{Iw}}^1} & H_{\text{Iw}}^1(\mathbb{Q}_p, V) \\ \downarrow 1-\varphi & & \downarrow (\underline{\text{Col}}_1, \underline{\text{Col}}_2) \\ (\varphi^* \mathbb{N}(V))^{\psi=0} & \xrightarrow{\mathfrak{J}} & \Lambda_{\mathbb{Q}_p}(G_\infty)^{\oplus 2} \\ \downarrow M & & \downarrow \underline{M} \\ ((\mathbb{B}_{\text{rig},\mathbb{Q}_p}^+)^{\psi=0})^{\oplus 2} & \xrightarrow{\mathfrak{M}^{-1}} & \mathcal{H}(G_\infty)^{\oplus 2} \\ \downarrow \text{pr}_i & & \downarrow \underline{\text{pr}}_i \\ (\mathbb{B}_{\text{rig},\mathbb{Q}_p}^+)^{\psi=0} & \xrightarrow{\mathfrak{M}^{-1}} & \mathcal{H}(G_\infty). \end{array}$$

$\mathcal{L}_{1,\bar{\nu}_i \otimes (1+\pi)}$

Here, pr_i and $\underline{\text{pr}}_i$ denote the projection maps onto the respective i -th components, and for an element $x \in (\varphi^* \mathbb{N}(V))^{\psi=0}$, $M.x$ is defined as follows: if $x = x_1 \varphi(\pi^{1-k} n_1 \otimes e_{k-1}) + x_2 \varphi(\pi^{1-k} n_2 \otimes e_{k-1})$ with $x_1, x_2 \in (\mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$, then $M.x = M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

3.5. Bounded p -adic L -functions. We now establish some basic properties of $L_{p,i}$ and $\tilde{L}_{p,i}$.

3.5.1. Decomposition of p -adic L -functions. Recall that α and β are the roots of the quadratic $X^2 - a_p X + p^{k-1}$. By [Kat04, Theorem 16.6], there exist eigenvectors η_α and η_β of φ in $E(\alpha) \otimes_{\mathbb{E}} \mathbb{D}_{\text{cris}}(V_f)$ with eigenvalues α and β respectively such that $[\eta_\alpha, \bar{\nu}_1] = [\eta_\beta, \bar{\nu}_1] = 1$ and we have

$$\begin{aligned} \langle \Omega_{V_f(1),1}((1+\pi) \otimes \eta_\alpha), \mathbf{z}^{\text{Kato}} \rangle_{V_f(1)} &= \tilde{L}_{p,\alpha}, \\ \langle \Omega_{V_f(1),1}((1+\pi) \otimes \eta_\beta), \mathbf{z}^{\text{Kato}} \rangle_{V_f(1)} &= \tilde{L}_{p,\beta}. \end{aligned}$$

It can be verified that

$$\begin{aligned} \eta_\alpha &= \alpha^{-1} \nu_1 - \nu_2, \\ \eta_\beta &= \beta^{-1} \nu_1 - \nu_2. \end{aligned}$$

Therefore, by Definition 3.17 and Proposition 3.22, we have

$$\begin{aligned} \mathfrak{M}(\tilde{L}_{p,\alpha}) &= (\alpha^{-1} m_{12} + m_{11}) L_{p,1} + (\alpha^{-1} m_{22} + m_{21}) L_{p,2}, \\ \mathfrak{M}(\tilde{L}_{p,\beta}) &= (\beta^{-1} m_{12} + m_{11}) L_{p,1} + (\beta^{-1} m_{22} + m_{21}) L_{p,2}; \end{aligned}$$

in the notation of Section 1.2, we have

$$\mathcal{M} = \begin{pmatrix} \alpha^{-1} m_{12} + m_{11} & \alpha^{-1} m_{22} + m_{21} \\ \beta^{-1} m_{12} + m_{11} & \beta^{-1} m_{22} + m_{21} \end{pmatrix}.$$

The functions $L_{p,1}$ and $L_{p,2}$ can therefore be written as

$$(30) \quad L_{p,1} = \frac{(\beta^{-1} m_{22} + m_{21}) \mathfrak{M}(\tilde{L}_{p,\alpha}) - (\alpha^{-1} m_{22} + m_{21}) \mathfrak{M}(\tilde{L}_{p,\beta})}{(\beta^{-1} - \alpha^{-1}) \det(M)}$$

$$(31) \quad L_{p,2} = \frac{(\beta^{-1} m_{12} + m_{11}) \mathfrak{M}(\tilde{L}_{p,\alpha}) - (\alpha^{-1} m_{12} + m_{11}) \mathfrak{M}(\tilde{L}_{p,\beta})}{(\alpha^{-1} - \beta^{-1}) \det(M)}$$

Let $x \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$. On the one hand,

$$(1 - \varphi)x = \underline{\text{Col}}_1(x) \cdot [(1 + \pi)\varphi(\pi^{1-k} n_1 \otimes e_{k-1})] + \underline{\text{Col}}_2(x) \cdot [(1 + \pi)\varphi(\pi^{1-k} n_2 \otimes e_{k-1})].$$

On the other hand, Proposition 3.22 says that

$$(1 - \varphi)x = \mathfrak{M} \circ \mathcal{L}_{1,\nu_1 \otimes (1+\pi)}(x) \bar{\nu}_2 \otimes t^{1-k} e_{k-1} - \mathfrak{M} \circ \mathcal{L}_{1,\nu_2 \otimes (1+\pi)}(x) \bar{\nu}_1 \otimes t^{1-k} e_{k-1}.$$

Therefore, we have

$$(\underline{\text{Col}}_1(x) \quad \underline{\text{Col}}_2(x)) \cdot [(1 + \pi)M] = (-\mathfrak{M} \circ \mathcal{L}_{1,\nu_2 \otimes (1+\pi)}(x) \quad \mathfrak{M} \circ \mathcal{L}_{1,\nu_1 \otimes (1+\pi)}(x)).$$

Let $\underline{M} = (\underline{m}_{ij}) = \mathfrak{M}^{-1}[(1 + \pi)M]$, then as elements of $\mathcal{H}(G_\infty)$

$$(32) \quad (\underline{\text{Col}}_1(x) \quad \underline{\text{Col}}_2(x)) \underline{M} = (-\mathcal{L}_{1,\nu_2 \otimes (1+\pi)}(x) \quad \mathcal{L}_{1,\nu_1 \otimes (1+\pi)}(x)).$$

Therefore, by exactly the same calculation as above, we have the following theorem:

Theorem 3.25. Define

$$\underline{\mathcal{M}} = \begin{pmatrix} \alpha^{-1} \underline{m}_{12} + \underline{m}_{11} & \alpha^{-1} \underline{m}_{22} + \underline{m}_{21} \\ \beta^{-1} \underline{m}_{12} + \underline{m}_{11} & \beta^{-1} \underline{m}_{22} + \underline{m}_{21} \end{pmatrix}.$$

Then we have the decomposition

$$(33) \quad \begin{pmatrix} \tilde{L}_{p,\alpha} \\ \tilde{L}_{p,\beta} \end{pmatrix} = \underline{\mathcal{M}} \begin{pmatrix} L_{p,1} \\ L_{p,2} \end{pmatrix}$$

Again, in the notation of Section 1.2, we have

$$\underline{\mathcal{M}} = \begin{pmatrix} \alpha^{-1} \underline{m}_{12} + \underline{m}_{11} & \alpha^{-1} \underline{m}_{22} + \underline{m}_{21} \\ \beta^{-1} \underline{m}_{12} + \underline{m}_{11} & \beta^{-1} \underline{m}_{22} + \underline{m}_{21} \end{pmatrix}.$$

3.5.2. *Interpolating properties.* We calculate the values of our new p -adic L -functions at characters modulo p . We first state a lemma concerning such characters.

Lemma 3.26. *If $A \in (\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0}$ is divisible by $\varphi(\pi)$, then $\mathfrak{M}^{-1}(A)$ is zero when evaluated at any character with conductor p .*

Proof. This is a special case of Theorem 5.4 as proved below. \square

Notation 3.27. *For any element $x \in \mathbb{C}_p \otimes (\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0}$ and η a character on G_∞ , we write $\eta(x)$ for $\eta(\mathfrak{M}^{-1}(x))$.*

Proposition 3.28. *Let η be a primitive character modulo p , then*

$$\begin{aligned} \eta(L_{p,1}) &= \frac{\tau(\eta)}{p^{k-1}} \cdot \frac{L(f_{\eta^{-1}}, 1)}{\Omega_f^{\eta(-1)}}, \\ \eta(L_{p,2}) &= 0. \end{aligned}$$

Similarly, if η is the trivial character, then

$$\begin{aligned} \eta(L_{p,1}) &= \frac{a_p - p^{k-2} - 1}{p^{k-1}} \cdot \frac{L(f, 1)}{\Omega_f^+}, \\ \eta(L_{p,2}) &= \left(\frac{1}{p} - 1\right) \cdot \frac{L(f, 1)}{\Omega_f^+}. \end{aligned}$$

Proof. Since

$$M = (t/\pi q)^{k-1} P^T M'^{-1} = (t/\pi q)^{k-1} \varphi(M'^{-1}) A_\varphi^T$$

and $M'|_{\pi=0} = I$, we have $M|_{\pi=(\zeta-1)} = A_\varphi^T$ for any p -th root of unity ζ . In other words, we have $M \equiv A_\varphi^T \pmod{\varphi(\pi)}$. Therefore, (30) and (31) imply that,

$$(34) \quad L_{p,1} \equiv \frac{(\beta^{-1} a_p - 1) \mathfrak{M}(\tilde{L}_{p,\alpha}) - (\alpha^{-1} a_p - 1) \mathfrak{M}(\tilde{L}_{p,\beta})}{(\beta^{-1} - \alpha^{-1}) p^{k-1}} \pmod{\varphi(\pi)}$$

$$(35) \quad L_{p,2} \equiv \frac{\beta^{-1} \mathfrak{M}(\tilde{L}_{p,\alpha}) - \alpha^{-1} \mathfrak{M}(\tilde{L}_{p,\beta})}{(\alpha^{-1} - \beta^{-1})} \pmod{\varphi(\pi)}$$

Therefore, we are done by Lemma 3.26 and the values of $\eta(\tilde{L}_{p,\alpha})$ and $\eta(\tilde{L}_{p,\beta})$ given in [MTT86] for example. \square

Corollary 3.29. *If $k \geq 3$, then $L_{p,i} \neq 0$ for $i \in \{1, 2\}$. Moreover, if η is a character of Δ , then $L_{p,1}^\eta \neq 0$.*

Proof. Since $k \geq 3$, the result follows from the fact that $L(f_{\eta^{-1}}, 1) \neq 0$ (by [Shi76, Proposition 2]). \square

Remark 3.30. *If $k = 2$ and $a_p = 0$, we will show that under the Mellin transform, $L_{p,1}$ and $L_{p,2}$ agree with Pollack's plus and minus p -adic L -functions up to a unit. Therefore, by [Pol03, Corollary 5.11], it is in fact enough to assume that assumption (A) holds in order for Corollary 3.29 to hold.*

Remark 3.31. *We see that the interpolating properties of $L_{p,1}$ and $L_{p,2}$ at a character modulo p are independent of the choice of n_1, n_2 as long as we have fixed a pair of 'good bases' for $\mathbb{D}_{\text{cris}}(V_f)$ and $\mathbb{D}_{\text{cris}}(V_{\bar{f}})$.*

Lemma 3.32. *If $z \in \Lambda_E(G_\infty)$ and $f \in (E \otimes \mathbb{B}_{\mathbb{Q}_p})^{\psi=0}$, then $z \cdot (f n_i) \equiv (z \cdot f) n_i \pmod{\varphi(\pi)}$ for $i = 1, 2$.*

Proof. It follows from the fact that if $g \in G_\infty$, $g(\varphi \mathbb{N}(V)) \subset \varphi(\pi) \mathbb{N}(V)$ for any V . \square

Corollary 3.33. *Proposition 3.28 (and hence Corollary 3.29) still hold after replacing $L_{p,i}$ by $\tilde{L}_{p,i}$ for $i = 1, 2$.*

Proof. By definitions, we have

$$(1 - \varphi)(\mathbf{z}^{\text{Kato}}) = (L_{p,1} \ L_{p,2}) M \begin{pmatrix} \bar{\nu}_1 \\ \bar{\nu}_2 \end{pmatrix} \otimes t^{1-k} e_{k-1} = (\tilde{L}_{p,1} \ \tilde{L}_{p,2}) \cdot \begin{pmatrix} (1 + \pi)n_1 \\ (1 + \pi)n_2 \end{pmatrix}$$

where \mathbf{z}^{Kato} is the localization of the Kato zeta element and M is as defined in (20). This implies

$$(L_{p,1} \ L_{p,2}) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = (\tilde{L}_{p,1} \ \tilde{L}_{p,2}) \cdot \begin{pmatrix} (1 + \pi)n_1 \\ (1 + \pi)n_2 \end{pmatrix}.$$

Therefore, by Lemma 3.32, we have $L_{p,i} \equiv \mathfrak{M}(\tilde{L}_{p,i}) \pmod{\varphi(\pi)}$ and hence $\mathfrak{M}^{-1}(L_{p,i})$ agrees with $\tilde{L}_{p,i}$ at a character modulo p by Lemma 3.26. \square

3.5.3. Infinitude of zeros. Let η be a character of Δ . Mazur proved that at least one of $\tilde{L}_{p,\alpha}$ and $\tilde{L}_{p,\beta}$ has infinitely many zeros if $v_p(\alpha) \neq v_p(\beta)$. This has been generalized to the case $a_p = 0$ ([Pol03, Theorem 3.5]). We show that our decomposition of $\tilde{L}_{p,\alpha}$ and $\tilde{L}_{p,\beta}$ can be used to give an alternative proof to Mazur's result as well as generalize Pollack's result to the case $a_p \neq 0$.

Proposition 3.34. *If f is a modular form as given in the beginning of Section 3.3 and η a character of Δ , then either $\tilde{L}_{p,\alpha}^\eta$ or $\tilde{L}_{p,\beta}^\eta$ has infinitely many zeros.*

Proof. Assume not, then [Pol03, Lemma 3.2] implies that $\tilde{L}_{p,\alpha}^\eta$ and $\tilde{L}_{p,\beta}^\eta$ are $O(1)$.

By [BB10, Lemmas 3.3.5 and 3.3.6], the entries of M are $O(\log_p^m)$ where $m = \max\{v_p(\alpha), v_p(\beta)\} < k - 1$. Therefore, with the notation above, $m_{ij} = O(\log_p^m)$ for $i, j \in \{1, 2\}$. In particular, the η -component of

$$(\beta^{-1}m_{22} + m_{21})\tilde{L}_{p,\alpha} - (\alpha^{-1}m_{22} + m_{21})\tilde{L}_{p,\beta}$$

is $O(\log_p)^m$. By (30), the quantity above is divisible by $(t/\pi q)^{k-1} \sim \log_p^{k-1}$ which forces $L_{p,1}^\eta = 0$ contradicting Corollary 3.28. \square

As with [Pol03, Theorem 3.5], we have:

Corollary 3.35. *If $\alpha \notin E(\eta)$, then both $\tilde{L}_{p,\alpha}^\eta$ and $\tilde{L}_{p,\beta}^\eta$ have infinitely many zeros.*

3.6. Good ordinary modular forms. We now assume that f is good ordinary at p . We will pick different bases from the supersingular case to define our Coleman maps. Let α be the root of $X^2 - a_p X + p^{k-1}$ which is a p -adic unit and β is the one with p -adic valuation $k - 1$. By a result of Deligne and Mazur-Wiles (see for example [Kat04, Section 17] for an exposition), there exists a 1-dimensional $G_{\mathbb{Q}_p}$ -subrepresentation V'_f in $V_{\bar{f}}$. Moreover, V'_f has Hodge-Tate weight 0 and $\mathbb{D}_{\text{cris}}(V'_f)$ can be identified with the α -eigenspace of φ in $\mathbb{D}_{\text{cris}}(V_{\bar{f}})$. We fix a nonzero element $\bar{\nu}_1 \in \mathbb{D}_{\text{cris}}(V'_f)$. Then, by (7), $n_1 = \bar{\nu}_1$ is a basis of $\mathbb{N}(V'_f)$ over $E \otimes \mathbb{B}_{\mathbb{Q}_p}^+$. Let $\bar{\nu}_2$ be a nonzero β -eigenvector of φ in $\mathbb{D}_{\text{cris}}(V_{\bar{f}})$.

Proposition 3.36. *We may find $n_2 \in \mathbb{N}(V_{\bar{f}})$ lifting $\bar{\nu}_2$ such that n_1, n_2 is an $E \otimes \mathbb{B}_{\mathbb{Q}_p}^+$ -basis of $\mathbb{N}(V_{\bar{f}})$, and $(1 + \pi)\varphi(\pi^{1-k}n_1 \otimes e_{k-1}), (1 + \pi)\varphi(\pi^{1-k}n_2 \otimes e_{k-1})$ is a $\Lambda_E(G_\infty)$ -basis of $(\varphi^* \mathbb{N}(V_{\bar{f}}(k-1)))^{\psi=0}$.*

Proof. Let $N = \mathbb{N}(V_{\bar{f}})$ and $N' = \mathbb{N}(V'_f)$. Then the quotient $N'' = N/N'$ may be identified with the Wach module of the quotient $V_{\bar{f}}/V'_f$, and we have an exact sequence

$$0 \longrightarrow (\varphi^* N'(k-1))^{\psi=0} \longrightarrow (\varphi^* N(k-1))^{\psi=0} \longrightarrow (\varphi^* N''(k-1))^{\psi=0} \longrightarrow 0.$$

It is clear that $(1 + \pi)\varphi(n_1 \otimes \pi^{1-k}e_{k-1})$ is a basis of $(\varphi^* N'(k-1))^{\psi=0}$, and the result now follows on applying theorem 3.5 to N'' . \square

Hence the change of basis matrix M' , with

$$\begin{pmatrix} \bar{\nu}_1 \\ \bar{\nu}_2 \end{pmatrix} = M' \begin{pmatrix} n_1 \\ n_2 \end{pmatrix},$$

is lower triangular, with $1, (t/\pi)^{k-1}$ on the diagonal. With respect to this basis, the Coleman maps given in Section 3.1 enable us to define:

Definition 3.37. For $i = 1, 2$, define $L_{p,i} \in (E \otimes \mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$ to be the image of the localization of the Kato zeta element (on using the identification as given by (4)) under Col_i . Similarly, define $\tilde{L}_{p,i}$ to be the image of the localization of the Kato zeta element under $\underline{\text{Col}}_i$.

Since $\varphi(n_1) = \alpha n_1$, the matrix P as defined in Section 3.1 is upper triangular and there exists a unit u in $E \otimes \mathbb{B}_{\mathbb{Q}_p}^+$ such that

$$P = \begin{pmatrix} \alpha & * \\ 0 & uq^{k-1} \end{pmatrix}.$$

Therefore, (10) becomes

$$(36) \quad (1 - \varphi)(x) = (\text{Col}_1(x) \quad \text{Col}_2(x)) \begin{pmatrix} \alpha(\frac{t}{\pi q})^{k-1} & 0 \\ * & u \end{pmatrix} \begin{pmatrix} \bar{\nu}_1 \\ \bar{\nu}_2 \end{pmatrix} \otimes t^{1-k} e_{k-1}.$$

Lemma 3.38. Let ν_1, ν_2 be a basis of $\mathbb{D}_{\text{cris}}(V_f)$ such that $\varphi(\nu_1) = \alpha \nu_1$ and $\varphi(\nu_2) = \beta \nu_2$. Then

$$[\nu_i \otimes t^{-1} e_1, \bar{\nu}_i \otimes t^{1-k} e_{k-1}] = 0$$

for $i = 1, 2$ where $[\cdot, \cdot]$ is the pairing defined in (19).

Proof. Assume $m_1 := [\nu_1 \otimes t^{-1} e_1, \bar{\nu}_1 \otimes t^{1-k} e_{k-1}] \neq 0$. Since $[\cdot, \cdot]$ is compatible with φ , we have

$$\begin{aligned} \varphi[\nu_1 \otimes t^{-1} e_1, \bar{\nu}_1 \otimes t^{1-k} e_{k-1}] &= [\varphi(\nu_1 \otimes t^{-1} e_1), \varphi(\bar{\nu}_1 \otimes t^{1-k} e_{k-1})] \\ p^{-1} m_1 &= [\alpha p^{-1} \nu_1 \otimes t^{-1} e_1, \alpha p^{1-k} \bar{\nu}_1 \otimes t^{1-k} e_{k-1}] \\ p^{k-1} m_1 &= \alpha^2 m_1. \end{aligned}$$

Hence, $\alpha^2 = p^{k-1}$, which is a contradiction. The proof for $i = 2$ is similar. \square

As in Section 3.4, we may assume that $[\nu_1, \bar{\nu}_2] = -[\nu_2, \bar{\nu}_1] = 1$ and an analogue of Proposition 3.22 says that

$$\mathfrak{M}(-\mathcal{L}_{\nu_2 \otimes (1+\pi)} \circ h_{\text{Iw}}^1 \quad \mathcal{L}_{\nu_1 \otimes (1+\pi)} \circ h_{\text{Iw}}^1) = (\text{Col}_1 \quad \text{Col}_2) \begin{pmatrix} \alpha(\frac{t}{\pi q})^{k-1} & 0 \\ * & u \end{pmatrix}.$$

In particular, if we apply this to the Kato zeta element, we have

$$(-\mathfrak{M}(\tilde{L}_{p,\beta}) \quad \mathfrak{M}(\tilde{L}_{p,\alpha})) = (L_{p,1} \quad L_{p,2}) \begin{pmatrix} \alpha(\frac{t}{\pi q})^{k-1} & 0 \\ * & u \end{pmatrix}$$

where $\tilde{L}_{p,\beta} = \mathcal{L}_{\nu_2}(\mathbf{z}^{\text{Kato}})$. Similarly, we have

$$(37) \quad (-\tilde{L}_{p,\beta} \quad \tilde{L}_{p,\alpha}) = (\tilde{L}_{p,1} \quad \tilde{L}_{p,2}) \begin{pmatrix} \alpha \log_{p,k} & 0 \\ * & \tilde{u} \end{pmatrix}$$

where $\log_{p,k} = \prod_{j=0}^{k-2} \log_p(\chi(\gamma)^{-j} \gamma) / (\chi(\gamma)^{-j} \gamma - 1)$ and $\tilde{u} \in \Lambda_E(G_\infty)^\times$.

Therefore, as in Section 3.4, we can decompose $\tilde{L}_{p,\beta}$ into a linear combination of $\tilde{L}_{p,1}$ and $\tilde{L}_{p,2}$, whereas $\tilde{L}_{p,\alpha} = \tilde{L}_{p,2}$, up to a unit. We now say something about $\tilde{L}_{p,1}$. When V_f is not locally split at p , $\tilde{L}_{p,\beta}$ is conjecturally equal to the critical slope p -adic L -function constructed in [PoS09]. We itemize this condition since we will need it again later.

• **Assumption (A'): V_f is not locally split at p and $k \geq 3$.**

In this case, [Kat04, Theorem 16.4 and 16.6] imply that $\tilde{L}_{p,\beta}$ has the same interpolating properties as $\tilde{L}_{p,\alpha}$, namely:

$$(38) \quad \chi^r \eta(\tilde{L}_{p,\alpha}) = \frac{c_{\eta,r}}{\beta^n} L(f_{\eta^{-1}}, r+1) \quad \text{and} \quad \chi^r \eta(\tilde{L}_{p,\beta}) = \frac{c_{\eta,r}}{\beta^n} L(f_{\eta^{-1}}, r+1)$$

where η is a finite character of conduction $p^n > 1$, $0 \leq r \leq k-2$ and $c_{\eta,r}$ is some constant independent of α and β . Note that the values given by (38) do not determine $\tilde{L}_{p,\beta}$ uniquely. However, they allow us to show that $\tilde{L}_{p,1}, L_{p,1} \neq 0$.

Proposition 3.39. If assumption (A') holds, then $\tilde{L}_{p,1}^\eta, L_{p,1}^\eta \neq 0$ for any character η on Δ .

Proof. As in the proof of Proposition 3.28, $M'|_{\pi=0}$ implies that $M|_{\pi=(\zeta-1)} = A_\varphi^T$ for any $\zeta^p = 1$, where $A_\varphi = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ is the matrix of φ with respect to $\bar{\nu}_1, \bar{\nu}_2$. Therefore, $\mathfrak{M}(\tilde{L}_{p,\beta})(\zeta-1) = \alpha L_{p,1}(\zeta-1)$. Since V_f is not locally split and $k \geq 3$, by the above discussion, $\eta(\tilde{L}_{p,\beta}) = \frac{\tau(\eta)}{\beta} L(f_{\eta-1}, 1) \neq 0$ as in the supersingular case. Therefore, $L_{p,1}(\zeta-1) \neq 0$. The statement about $\tilde{L}_{p,1}$ then follows as in Corollary 3.33. \square

We see from the proof that the interpolating properties of $\mathfrak{M}^{-1}(L_{p,1})$ and $\tilde{L}_{p,1}$ at characters modulo p are the same as that of $\tilde{L}_{p,\beta}$ after multiplying a constant.

Remark 3.40. *If V_f does split locally at p , we can choose $n_2 = \bar{\nu}_2$ and both P and M' would be diagonal. Therefore, we have $\tilde{L}_{p,\beta} = \mathfrak{M}^{-1}((t/\pi q)^{k-1} L_{p,1}) = \log_{p,k} \tilde{L}_{p,1}$. But it is not known that whether $\tilde{L}_{p,\beta}$ is nonzero or not.*

4. COLEMAN MAPS FOR THE BERGER–LI–ZHU BASIS

In this section, we will prove some results on the images of the Coleman maps under the assumption that $v_p(a_p) > \lfloor \frac{k-2}{p-1} \rfloor$, using the explicit basis of $\mathbb{N}(V_f)$ written down in [BLZ04]. We shall also give an explicit proof that this particular basis satisfies the conclusions of theorem 3.15.

Write $m = \lfloor (k-2)/(p-1) \rfloor$ and define

$$\log^+(1+\pi) = \prod_{n \geq 0} \frac{\varphi^{2n+1}(q)}{p} = \prod_{\substack{n \geq 1 \\ n \text{ even}}} \frac{\Phi_n(1+\pi)}{p}$$

and

$$\text{and } \log^-(1+\pi) = \prod_{n \geq 0} \frac{\varphi^{2n}(q)}{p} = \prod_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\Phi_n(1+\pi)}{p}.$$

where $\Phi_n(X)$ is the p^n -th cyclotomic polynomial. Let z_i be elements of \mathbb{Q}_p such that

$$p^m \left(\frac{\log^-(1+\pi)}{\log^+(1+\pi)} \right)^{k-1} = \sum_{i \geq 0} z_i \pi^i,$$

then as shown in [BLZ04, Proposition 3.1.1],

$$z = \sum_{i=0}^{k-2} z_i \pi^i \in \mathbb{Z}_p[[\pi]].$$

By [BLZ04], under assumption (C), i.e. $v_p(a_p) > m$, there is a lattice $T_{\bar{f}}$ in $V_{\bar{f}}$ and a basis of $\mathbb{N}(T_{\bar{f}})$ such that the matrix of φ with respect to this basis, P , is given by

$$\begin{pmatrix} 0 & -1 \\ q^{k-1} & \delta z \end{pmatrix}$$

where $\delta = a_p/p^m$. In particular, the reduction of this basis modulo π is a “good basis” in the sense of §3.3, and hence the Coleman maps may be defined integrally as in remark 3.16. By construction, for any $x \in \mathbb{D}(T_{\bar{f}}(k-1))^{\psi=1}$ with

$$x = \pi^{1-k} (x_1 \ x_2) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1},$$

we can express $\text{Col}_i(x)$, $i = 1, 2$, in terms of x_1 and x_2 :

$$(39) \quad \text{Col}_1(x) = x_2 - \varphi(x_1) + \delta z x_1,$$

$$(40) \quad \text{Col}_2(x) = -q^{k-1} x_1 - \varphi(x_2).$$

Remark 4.1. *The representation constructed in [BLZ04] is really $V_{\bar{f}}$ twisted by an unramified character. But since we assume that $\epsilon(p) = 1$, it does not affect the action of P and our calculations later on.*

4.1. **The image of Col_1 .** We first give a few preliminary lemmas.

Lemma 4.2. *For all $n \geq 0$, we have $\varphi^n(M'^{-1})(A_\varphi^T)^n = \varphi^{n-1}(P^T) \cdots \varphi(P^T) P^T M'^{-1}$. Moreover, as $n \rightarrow \infty$, the quantity above tends to 0.*

Proof. The equality follows from (9) and induction. For the limit, note that $M'|_{\pi=0} = I$, hence $\varphi^n(M') \rightarrow I$ as $n \rightarrow \infty$. The eigenvalues of A_φ are α and β . But $\alpha^n, \beta^n \rightarrow 0$ as $n \rightarrow \infty$, so we are done. \square

Lemma 4.3. *Let $x = \pi^{1-k} (x_1 \ x_2) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1}$. Then, $\psi(x)$ is given by*

$$(\psi(x_1 \delta z + x_2) \ -\psi(q^{k-1} x_1)) \pi^{1-k} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Proof. Recall that $\varphi(\pi) = \pi q$, we have

$$\begin{aligned} x &= \pi^{1-k} (x_1 \ x_2) (P^T)^{-1} \begin{pmatrix} \varphi(n_1) \\ \varphi(n_2) \end{pmatrix} \\ &= (x_1 \delta z + x_2 \ -q^{k-1} x_1) \varphi(\pi)^{1-k} \begin{pmatrix} \varphi(n_1) \\ \varphi(n_2) \end{pmatrix}, \end{aligned}$$

hence the result \square

Lemma 4.4. *For all $n \geq 1$, the constant term of $\psi(q^n)$ is p^{n-1} .*

Proof. Induction. \square

Lemma 4.5. *If $f(\pi) \in E \otimes \mathbb{B}_{\mathbb{Q}_p}^+$, then there exist unique $a_i \in E$ for $1 \leq i \leq k-1$ such that $f(\pi) = \sum_{i=1}^{k-1} a_i(\pi+1)^i \pmod{\pi^{k-1}}$.*

Proof. Note that

$$(41) \quad (\pi+1)^k = \binom{k}{1}(\pi+1)^{k-1} - \cdots + (-1)^{k-2} \binom{k}{k-1}(\pi+1) + (-1)^{k-1} \pmod{\pi^k}.$$

Suppose now that there exist $a_1, \dots, a_{k-1} \in E$ such that $(\pi+1)^k = \sum_{i=1}^{k-1} a_i(\pi+1)^i \pmod{\pi^k}$. Subtracting this sum from (41) shows that

$$(\binom{k}{1} - a_{k-1})(\pi+1)^{k-1} + \cdots + ((-1)^{k-2} \binom{k}{k-1} - a_1)(\pi+1) + (-1)^{k-1} = 0.$$

But this gives a contradiction since $\{(\pi+1)^i\}_{0 \leq i < k}$ is a basis of the vector space of polynomials of degree $\leq k-1$. \square

Proposition 4.6. *Under assumption (C), the map $(\pi^{k-1} \mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0} \subset \text{Col}_1(\mathbb{D}(T_{\bar{f}}(k-1))^{\psi=1})$.*

Proof. Recall that (6) says

$$(1 - \varphi)x = (y_1 \ y_2) \cdot (\pi q)^{1-k} P^T \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1}.$$

For any $y_1 \in (\pi^{k-1} \mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$, we have

$$y := (y_1 \ 0) \cdot (\pi q)^{1-k} P^T \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1} = (0 \ y_1/\pi^{k-1}) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1}.$$

Then,

$$\begin{aligned} \varphi^n(y) &= (0 \ \varphi^n(y_1/\pi^{k-1})) \varphi^{n-1}(P^T) \cdots \varphi(P^T) P^T \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1} \\ &= (0 \ \varphi^n(y_1/\pi^{k-1})) \varphi^n(M'^{-1})(A_\varphi^T)^n M' \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1}. \end{aligned}$$

Hence, Lemma 4.2 implies that $\varphi^n(y) \rightarrow 0$ as $n \rightarrow \infty$ and the series $x := \sum_{n \geq 0} \varphi^n(y)$ converges to an element of $\mathbb{D}(T_{\bar{f}}(k-1))^{\psi=1}$ with $(1 - \varphi)x = y$. Therefore, $y_1 = \text{Col}_1(x)$. \square

Proposition 4.7. *Under assumptions (B), (C) and (D), the map $\text{Col}_1 : \mathbb{D}(V_{\bar{f}}(k-1)) \rightarrow (E \otimes \mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$ is surjective.*

Proof. By Proposition 4.6, if $y_1 \in (\pi^{k-1} E \otimes \mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$, then $y_1 \in \text{Im}(\text{Col}_1)$. For an arbitrary $y_1 \in (E \otimes \mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$, by Lemma 4.5 there exists y' in the E -linear span of $\{(1+\pi)^i\}_{1 \leq i < k}$ such that $y_1 + \varphi(y')$ is divisible by π^{k-1} . Hence, by the same argument as above, the sum

$$\sum_{n \geq 0} \varphi^n \left((0 \quad (y_1 + \varphi(y'))/\pi^{k-1}) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \right)$$

converges to an element $x \in \mathbb{N}(V_{\bar{f}}(k-1))$. By Lemma 4.3 and the fact that $\psi(y_1) = 0$, we have

$$\begin{aligned} \psi(x) - x &= \psi \left((0 \quad (y_1 + \varphi(y'))/\pi^{k-1}) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \right) \\ &= \pi^{1-k} (y' \quad 0) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \end{aligned}$$

Let $x' = x + \pi^{1-k} (x_1 \quad x_2) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$. Then

$$\psi(x') - x' = \pi^{1-k} (y' - x_1 + \psi(x_1 \delta z + x_2) \quad -x_2 - \psi(q^{k-1} x_1)) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}.$$

Hence, $x' \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$ if and only if

$$(42) \quad \begin{aligned} x_2 &= -\psi(q^{k-1} x_1) \\ y' &= x_1 - \psi(x_1 \delta z) + \psi^2(q^{k-1} x_1) \end{aligned}$$

Assume that such x_1 exists in the E -linear span of $\{(1+\pi)^i\}_{1 \leq i < k}$, and let a be its degree in π . Since the degrees of δz and q^{k-1} are at most $k-2$ and $(p-1)(k-1)$ respectively, the degrees of $\psi(x_1 \delta z)$ and $\psi^2(q^{k-1} x_1)$ are at most $(k-2+a)/p$ and $((p-1)(k-1)+a)/p^2$ respectively. But we assume that $p \geq k-1$, so the right-hand side of (42) has degree $\leq a$. Since y' has degree at most $k-1$ and x_1 is in the E -linear span of $\{(1+\pi)^i\}_{1 \leq i < k}$, both $\psi(x_1 \delta z)$ and $\psi^2(q^{k-1} x_1)$ are scalar multiples of $(1+\pi)$. We write

$$y' = \sum_{i=1}^{k-1} \alpha_i (1+\pi)^i, \quad x_1 = \sum_{i=1}^{k-1} \beta_i (1+\pi)^i \quad \text{and} \quad \delta z = \sum_{i=0}^{k-2} \gamma_i (1+\pi)^i$$

where $\alpha_i, \beta_i, \gamma_i \in E$. Then (42) says that

$$\begin{aligned} \alpha_i &= \beta_i \quad \text{for } i \geq 2 \\ \alpha_1 &= \beta_1 - \sum_{i+j=p} \beta_i \gamma_j + \beta_{p^2-(k-1)(p-1)} \end{aligned}$$

where $\gamma_i = \beta_i = 0$ if $i < 0$. But $p^2 - (k-1)(p-1) > 1$ and $\gamma_{p-1} = 0$, the matrix relating $(\alpha_i)_{1 \leq i \leq k-1}$ and $(\beta_i)_{1 \leq i \leq k-1}$ is upper triangular with non-zero entries on the diagonal. Therefore, there is a bijection between $(\alpha_i)_{1 \leq i \leq k-1} \in E^{k-1}$ and $(\beta_i)_{1 \leq i \leq k-1} \in E^{k-1}$. In other words, given any y' as above, there exists a unique x_1 (and hence x_2) such that $x' \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$. For any $0 \leq j \leq k-2$, we can therefore choose y (and hence y') such that $x_1 \equiv \pi^j \pmod{\pi^{j+1}}$. In this case,

$$\begin{aligned} \text{Col}_1(x') &= y_1 + \varphi(y') - \psi(q^{k-1} x_1) - \varphi(x_1) + x_1 \delta z \\ &\equiv -\psi(q^{k-1} x_1) - \varphi(x_1) + x_1 \delta z \pmod{\pi^{k-1}} \\ &\equiv (-p^{k-2-j} - p^j + a_p) \pi^j \pmod{\pi^{j+1}}, \end{aligned}$$

where we deduce the last line from the previous one using Lemma 4.4 and the observation that $\pi q = \varphi(\pi)$. Therefore, our assumption on a_p implies that for all $y_1 \in (E \otimes \mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$, there exists some $x \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$ such that $\text{Col}^+(x) \equiv y_1 \pmod{\pi^{j+1}}$ by induction. Hence we are done. \square

Corollary 4.8. *Under assumptions (B), (C) and (D), the image of $\text{Col}_1 : \mathbb{D}(T_{\bar{f}}(k-1))^{\psi=1} \rightarrow (\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$ is pseudo isomorphic to $(\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$.*

Proof. It suffices to show that the said image has finite index in $(\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$. The proof of Proposition 4.6 shows that $(\pi^{k-1} \mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$ lies in the image and for all $0 \leq j \leq k-2$, there exists $x_j \in \mathbb{D}(T_{\bar{f}}(k-1))^{\psi=1}$ such that $\text{Col}_1(x_j) \equiv \alpha_j \pi^j \pmod{\pi^{j+1}}$ for some $\alpha_j \neq 0$. Therefore, the quotient lies inside $\prod_{j=0}^{k-2} \mathcal{O}_E / \alpha_j \mathcal{O}_E$, so we are done. \square

4.2. The image of Col_2 . We now describe the image of Col_2 . We will show that it is generated by two elements.

Lemma 4.9. *Let $x = \pi^{1-k} (x_1 \ x_2) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1}$ and γ a topological generator of Γ , then*

$$\gamma(x) = \pi^{1-k} (\gamma(x_1) \ \gamma(x_2)) G_\gamma \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1}$$

for some $G_\gamma \in I + \pi M(2, \mathbb{Z}_p[[\pi]])$.

Proof. By [BLZ04, Proposition 3.1.3], there exists $G_\gamma \in I + \pi M_2(\mathbb{Z}_p[[\pi]])$ such that $\begin{pmatrix} \gamma(n_1) \\ \gamma(n_2) \end{pmatrix} = G_\gamma^T \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$. Therefore,

$$\begin{aligned} \gamma(x) &= \gamma(\pi)^{1-k} (\gamma(x_1) \ \gamma(x_2)) G_\gamma^T \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes \chi(\gamma)^{k-1} e_{k-1} \\ &= \left(\frac{(1+\pi)^{\chi(\gamma)} - 1}{\chi(\gamma)} \right)^{1-k} (\gamma(x_1) \ \gamma(x_2)) G_\gamma^T \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1}. \end{aligned}$$

But $\chi(\gamma) \in 1 + p\mathbb{Z}_p$, which implies $((1+\pi)^{\chi(\gamma)} - 1)/\chi(\gamma) \in \pi(1 + p\mathbb{Z}_p[[\pi]])$. Hence the result. \square

Lemma 4.10. *Let $x = \pi^{1-k} (x_1 \ x_2) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1} \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$. Write $x_i = \sum_{j \geq 0} a_{i,j} \pi^j$. Then x_1 has order $< k-1$ if and only if x_2 has order $< k-1$. If this is the case, they have the same order which we denote by d_x . Moreover, $a_{2,d_x} = -p^{k-2-d_x} a_{1,d_x}$.*

Proof. Since $x \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$, we have $x_2 = -\psi(q^{k-1} x_1)$, hence the result by Lemma 4.4. \square

Proposition 4.11. *Under assumptions (C) and (D), the image of $\text{Col}_2 : \mathbb{D}(V_{\bar{f}}(k-1)) \rightarrow (E \otimes \mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$ contains $(\varphi(\pi)^{k-1} E \otimes \mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$ and the quotient of the containment is a cyclic $\Lambda_E(\Gamma)$ -module under the action of Γ described in Lemma 4.9.*

Proof. For any $y_2 \in (\varphi(\pi)^{k-1} E \otimes \mathbb{B}_{\mathbb{Q}_p}^+)^{\psi=0}$, we have

$$y := (0 \ y_2) \cdot (\pi q)^{1-k} P^T \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1} = \varphi(\pi)^{1-k} (-y_2 \ y_2 \delta z) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1}.$$

Hence, as in the proof of Proposition 4.6, $\sum_{n \geq 0} \varphi^n(y)$ converges which implies that y_2 lies in the image of Col_2 .

Recall that if $x = \pi^{1-k} (x_1 \ x_2) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1}$, then $-\text{Col}_2(x) = q^{k-1} x_1 + \psi(x_2)$. For $i = 1, 2$, write $x_i = \sum_{j \geq 0} a_{i,j} \pi^j$ and

$$\begin{aligned} \bar{\mathcal{C}}(x) &= q^{k-1} x_1 - \varphi(x_2) \pmod{\varphi(\pi)^{k-1}} \\ &= (q^{k-1} a_{1,0} + a_{2,0}) + \varphi(\pi)(q^{k-2} a_{1,1} + a_{2,1}) + \cdots + \varphi(\pi)^{k-2} (q a_{1,k-2} + a_{2,k-2}) \pmod{\varphi(\pi)^{k-1}}. \end{aligned}$$

We now construct a generator f for $\bar{\mathcal{C}}(\mathbb{D}(V)^{\psi=1})$ over $\Lambda_E(\Gamma)$ inductively. By the proof of Proposition 4.6, there exists $x_i \in \mathbb{D}(V)^{\psi=1}$ of order i for all $0 \leq i < k-1$. Let $f_0 = x_0$. For $i \geq 0$, suppose that we have constructed f_i . Write

$$\begin{aligned} f'_i &= \prod_{j=0}^i (\gamma - \chi(\gamma)^j)(f_i) \\ &= \pi^{1-k} (f'_{i,1} \quad f'_{i,2}) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes e_{k-1} \end{aligned}$$

then it follows from Lemma 4.9 that f'_i is of order $\geq i+1$. Let $\alpha_{i+1,1}$ and $\alpha_{i+1,2}$ be the coefficients of π^{i+1} in the power series expansions of $f'_{i,1}$ and $f'_{i,2}$, respectively. There are two possibilities: either both $\alpha_{i+1,j}$ are non-zero, in which case we let $f_{i+1} = f_i$. Or both of them are zero, in which case we let $f_{i+1} = f_i + x_{i+1}$.

Let $f = f_{k-2}$. Then for all $0 \leq i < k-1$, the order of $\prod_{j=0}^i (\gamma - \chi(\gamma)^j)(f)$ is i . To finish the proof, it is now sufficient to observe that by Lemma 4.10, for all $x \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$ there exist scalars $\alpha_i \in E$ for $0 \leq i < k-1$ such that $x - \sum_{i=1}^{k-2} \alpha_i \prod_{j=0}^i (\gamma - \chi(\gamma)^j)f$ is of order $\geq k-1$. \square

4.3. The Iwasawa transform.

Convention 4.12. *For the rest of this section as well as in Sections 4.4 and 4.5, we assume without loss of generality that $\chi(\gamma) = 1 + p$.*

Lemma 4.13.

$$(43) \quad \frac{q}{\gamma(q)} = 1 \pmod{(p\pi, \pi^{p-1})}.$$

Proof. We have $q = \frac{\varphi(\pi)}{\pi}$, and $\gamma(1 + \pi) = (1 + \pi)(1 + \varphi(\pi))$. Hence

$$\frac{q}{\gamma(q)} = \frac{1 + q + \varphi(\pi)}{1 + \varphi(q) + \varphi^2(\pi)}$$

It remains to notice that $q = \pi^{p-1} \pmod{p}$. Moreover, the constant term of q (and hence of $\varphi(q)$) is p , and $\sum_{j=0}^{+\infty} (-p)^j$ is the multiplicative inverse of $1 + p$, which implies the result. \square

Corollary 4.14. *Both $\frac{\log^+}{\gamma(\log^+)}$ and $\frac{\log^-}{\gamma(\log^-)}$ are congruent to $1 \pmod{(p\pi, \pi^{p-1})}$ (and hence in particular congruent to $1 \pmod{(p\pi, \pi^2)}$ since we assume $p \geq 3$).*

Proof. Clear from Lemma 4.13 and the definition of \log^{\pm} . \square

Define

$$G_{\gamma}^{(k-1)} = \begin{pmatrix} \left(\frac{\log^+}{\gamma(\log^+)}\right)^{k-1} & 0 \\ 0 & \left(\frac{\log^-}{\gamma(\log^-)}\right)^{k-1} \end{pmatrix}.$$

Lemma 4.15. $G_{\gamma}^{(k-1)} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{(p\pi, \pi^2)}.$

Proof. Immediate from the definition and Corollary 4.14. \square

Let ϖ_E be a uniformizer of E .

Proposition 4.16. $G_{\gamma} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{(\varpi_E \pi, \pi^2)}.$

Proof. We first review the construction of G_{γ} as in [BLZ04, §3.1]. For $l \geq k$, we define recursively

$$G_{\gamma}^{(l)} = G_{\gamma}^{(l-1)} + \pi^{l-1} H^{(l)}$$

for some $H^{(l)} \in M(2, \mathbb{Z}_p[[X]])$ where $X = a_p/p^m$ and $m = \lfloor \frac{k-2}{p-1} \rfloor$. Note that $X \in \mathfrak{m}_E$ by assumption (C). The matrix G_γ is then given by the limit of $G_\gamma^{(l)}$ as $l \rightarrow \infty$. Therefore, when $k > 2$, the result is immediate from Lemma 4.15.

When $k = 2$, it suffices to show that $H^{(2)} \equiv 0 \pmod{\varpi_E}$. By construction (see [BLZ04, Lemma 3.1.2 and Proposition 3.1.3]), $H^{(2)}$ satisfies the following:

$$(44) \quad H^{(2)} - P_0 H^{(2)} (pP_0)^{-1} = -R^{(1)} \pmod{\pi}$$

for some matrix $R^{(1)} \in XM(2, \mathbb{Z}_p[[\pi, X]])$ and $P_0 = \begin{pmatrix} 0 & 1 \\ p & a_p \end{pmatrix}$. If we write $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$, then (44) says that

$$\begin{pmatrix} h_{11} & h_{12} + h_{21} \\ h_{21} & h_{22} \end{pmatrix} \equiv 0 \pmod{X},$$

and hence we are done since $\varpi_E | X$. \square

Let $n'_i = \varphi(n_i \otimes \pi^{1-k} e_{k-1})$ for $i = 1, 2$. Let $T = T_{\bar{f}}(k-1)$ and $V = V_{\bar{f}}(k-1)$. (In fact, the proof works for $T = T_{\bar{f}}(m)$ for any integer m .) Recall that $\chi(\gamma) = 1 + p$.

Proposition 4.17. *We have $\gamma[(1 + \pi)n'_i] = (1 + \varphi(\pi))(1 + \pi)n'_i \pmod{(\varpi_E \varphi(\pi), \varphi(\pi)^2)}$ for $i = 1, 2$.*

Proof. We know that $\begin{pmatrix} n'_1 \\ n'_2 \end{pmatrix} = P^T \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \otimes \varphi(\pi)^{1-k} e_{k-1}$. Since the actions of γ and φ commute, we have $\gamma(P^T)G_\gamma^T = \varphi(G_\gamma^T)P^T$, which implies

$$\begin{pmatrix} \gamma n'_1 \\ \gamma n'_2 \end{pmatrix} = \chi(\gamma)^{k-1} \varphi \left(\frac{\pi}{\gamma(\pi)} \right)^{k-1} \varphi(G_\gamma^T) \begin{pmatrix} n'_1 \\ n'_2 \end{pmatrix}.$$

Now

$$(45) \quad \begin{aligned} \chi(\gamma) \frac{\pi}{\gamma(\pi)} &= \frac{\chi(\gamma)}{1 + q + \varphi(\pi)} \\ &\equiv 1 \pmod{(p\pi, \pi^2)} \end{aligned}$$

where the congruence comes from the fact that the constant term of q is p , and hence the constant term of $\frac{1}{1+q+\varphi(\pi)}$ is $\sum_{j=0}^{+\infty} (-p)^j$, which is equal to $\chi(\gamma)^{-1}$. Hence $\varphi \left(\frac{\chi(\gamma)\pi}{\gamma(\pi)} \right) \equiv 1 \pmod{(p\varphi(\pi), \varphi(\pi)^2)}$. Moreover, $\gamma(1 + \pi) = (1 + \pi)^{\chi(\gamma)} = (1 + \pi)(1 + \varphi(\pi))$. Hence

$$\gamma[(1 + \pi)n'_1] \equiv (1 + \varphi(\pi))(1 + \pi)n'_1 \pmod{(\varpi_E \varphi(\pi), \varphi(\pi)^2)}$$

by Corollary 4.16 \square

We will now show that we can adapt the arguments from Proposition 4.17 to pass from $\varphi(\pi)(1 + \pi)n'_i$ to $\varphi(\pi)^2(1 + \pi)n'_i \pmod{(\varpi_E \varphi(\pi), \varphi(\pi)^3)}$ for $i = 1, 2$.

Lemma 4.18. *We have $(\gamma - 1)[\varphi(\pi)(1 + \pi)n'_i] = \varphi(\pi)^2(1 + \pi)n'_i \pmod{(\varpi_E \varphi(\pi), \varphi(\pi)^3)}$ for $i = 1, 2$.*

Proof. We have $\gamma(\pi) = (1 + \pi)(1 + \varphi(\pi)) - 1 = \pi + \varphi(\pi) + \pi\varphi(\pi)$, so

$$\begin{aligned} \varphi(\gamma(\pi)) &= \varphi(\pi)(1 + \varphi(q) + \varphi(\pi)\varphi(q)) \\ &\equiv \varphi(\pi) \pmod{(\varpi_E \varphi(\pi), \varphi(\pi)^3)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \gamma[\varphi(\pi)(1 + \pi)n'_1] &\equiv \varphi(\pi)(1 + \varphi(q) + \varphi(\pi)\varphi(q))(1 + \pi)(1 + \varphi(\pi))n'_1 \pmod{(\varpi_E \varphi(\pi)^2, \varphi(\pi)^3)} \\ &\equiv \varphi(\pi)(1 + \pi)(1 + \varphi(\pi))n'_1 \pmod{(\varpi_E \varphi(\pi), \varphi(\pi)^3)} \end{aligned}$$

and hence

$$(46) \quad (\gamma - 1)[\varphi(\pi)(1 + \pi)n'_1] \equiv \varphi(\pi)^2(1 + \pi)n'_1 \pmod{(\varpi_E \varphi(\pi), \varphi(\pi)^3)}.$$

\square

The lemma generalizes as follows for arbitrary $r \geq 1$.

Proposition 4.19. *We have $(\gamma - 1)[\varphi(\pi)^r(1 + \pi)n'_i] \equiv \varphi(\pi)^{r+1}(1 + \pi)n'_i \pmod{(\varpi_E\varphi(\pi)^r, \varphi(\pi)^{r+2})}$ for $i = 1, 2$.*

Proof. Be the same calculations as in Lemma 4.18, we have

$$\begin{aligned} \gamma[\varphi(\pi)^r(1 + \pi)n'_1] &\equiv \varphi(\pi)^r(1 + \varphi(q) + \varphi(\pi)\varphi(q))^r(1 + \pi)(1 + \varphi(\pi))n'_1 \pmod{(\varpi_E\varphi(\pi)^{r+1}, \varphi(\pi)^{r+2})} \\ &\equiv \varphi(\pi)^r(1 + \pi)(1 + \varphi(\pi))n'_1 \pmod{(\varpi_E\varphi(\pi)^r, \varphi(\pi)^{r+2})} \end{aligned}$$

□

Definition 4.20. *For all $r \geq 2$, denote by I_r the ideal of $\varphi(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)$ generated by the elements*

$$\varpi_E^{r-1}\varphi(\pi), \varpi_E^{r-2}\varphi(\pi)^2, \dots, \varpi_E\varphi(\pi)^{r-1}, \varphi(\pi)^{r+1},$$

and let $\mathfrak{I}_r = I_r(\varphi^*\mathbb{N}(T))^{\psi=0}$.

Note that \mathfrak{I}_r is stable under the action of G_∞ .

Lemma 4.21. *We have $(\gamma - 1)\mathfrak{I}_r \subset \mathfrak{I}_{r+1}$.*

Proof. It is enough to show that $(\gamma - 1)[x\varphi(\pi)^m(1 + \pi)n'_i] \in \mathfrak{I}_{r+1}$ for any $m \geq 0$, any $x \in I_r$ and $i = 1, 2$.

Let $x = \varpi_E^{r-j}\varphi(\pi)^j$ where $1 \leq j \leq r - 1$. By Proposition 4.19, we have

$$\begin{aligned} (\gamma - 1)[\varpi_E^{r-j}\varphi(\pi)^{m+j}(1 + \pi)n'_i] &\equiv \varpi_E^{r-j}\varphi(\pi)^{m+j+1}(1 + \pi)n'_i \pmod{(\varpi_E^{r-j+1}\varphi(\pi)^{m+j}, \varpi_E^{r-j}\varphi(\pi)^{m+j+2})} \\ &\equiv 0 \pmod{\mathfrak{I}_{r+1}} \end{aligned}$$

for all $m \geq 0$. Similarly, the same holds for $x = \varphi(\pi)^{r+1}$. Hence the result. □

Proposition 4.22. *We have*

$$(\gamma - 1)^r[(1 + \pi)n'_i] \equiv \varphi(\pi)^r(1 + \pi)n'_i \pmod{\mathfrak{I}_r}$$

for all $r \geq 2$.

Proof. We proceed by induction on r . Let $r = 2$. By Proposition 4.17, we have

$$(\gamma - 1)[(1 + \pi)n'_i] \equiv \varphi(\pi)(1 + \pi)n'_i \pmod{(\varpi_E\varphi(\pi), \varphi(\pi)^2)}.$$

It therefore follows from Lemma 4.18 and Proposition 4.19 that

$$(\gamma - 1)^2[(1 + \pi)n'_i] \equiv \varphi(\pi)^2(1 + \pi)n'_i \pmod{(\varpi_E\varphi(\pi), \varphi(\pi)^3)}.$$

Assume now that the result is true for $r - 1 \geq 2$, so

$$(\gamma - 1)^{r-1}[(1 + \pi)n'_i] \equiv \varphi(\pi)^{r-1}(1 + \pi)n'_i \pmod{\mathfrak{I}_{r-1}}.$$

Now

$$\begin{aligned} (\gamma - 1)[\varphi(\pi)^{r-1}(1 + \pi)n'_i] &\equiv \varphi(\pi)^r(1 + \pi)n'_i \pmod{(\varpi_E\varphi(\pi)^{r-1}, \varphi(\pi)^{r+1})} \\ &\equiv \varphi(\pi)^r(1 + \pi)n'_i \pmod{\mathfrak{I}_r} \end{aligned}$$

by Proposition 4.19. The result therefore follows from Lemma 4.21. □

To simplify the notation, let $X = \varphi(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)(1 + \pi)n'_1 + \varphi(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)(1 + \pi)n'_2$.

Corollary 4.23. *For all $x \in X$, there exist $\omega_1, \omega_2 \in \Lambda_{\mathcal{O}_E}(\Gamma)$ such that*

$$\omega_1((1 + \pi)n'_1) + \omega_2((1 + \pi)n'_2) - x \in \varpi_E X.$$

Proof. $(\varphi^*\mathbb{N}(T))^{\psi=0}$ is complete in the $(\varpi_E, \varphi(\pi))$ -adic topology, and the \mathfrak{I}_r , $r \geq 1$ form a neighbourhood of zero in $(\varphi^*\mathbb{N}(T))^{\psi=0}$. Hence the result follows from Proposition 4.22. □

Note that $(\varphi^*\mathbb{N}(T))^{\psi=0}$ is the Δ -orbit of X . The previous corollary therefore implies the following result:

Theorem 4.24. $(\varphi^* \mathbb{N}(T))^{\psi=0}$ is a free $\Lambda_{\mathcal{O}_K}(G_\infty)$ -module of rank 2, and a basis is given by $(1 + \pi)n'_1$ and $(1 + \pi)n'_2$.

Proof. Let $y \in (\varphi^* \mathbb{N}(T))^{\psi=0}$. It follows from Corollary 4.23 and the fact that $\Lambda_{\mathcal{O}_E}(G_\infty)$ is p -adically complete that there exists $\omega_1, \omega_2 \in \Lambda_E(G_\infty)$ such that $y = \omega_1((1 + \pi)n'_1) + \omega_2((1 + \pi)n'_2)$. As shown in [PR94], $\mathbb{N}(T)^{\psi=1}$ is a free $\Lambda_E(G_\infty)$ -module of rank 2, and the map $1 - \varphi : \mathbb{N}(T)^{\psi=1} \rightarrow (\varphi^* \mathbb{N}(T))^{\psi=0}$ is injective since $V^{H_{\mathbb{Q}_p}} = \{0\}$. Hence the result. \square

It therefore follows that after tensoring with \mathbb{Q} , there is an isomorphism of $\Lambda_E(G_\infty)$ -modules (the *Iwasawa transform*)

$$\mathfrak{J} : (\varphi^* \mathbb{N}(V))^{\psi=0} \longrightarrow \Lambda_E(G_\infty)^{\oplus 2}$$

which satisfies the following condition: if $y = y_1(1 + \pi)n'_1 + y_2(1 + \pi)n'_2$ with $y_i \in \varphi(E \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathbb{Q}_p}^+)$ (write $y = (y_1, y_2)$) and $(z_1, z_2) = \mathfrak{J}(y_1, y_2)$, then $y = z_1[(1 + \pi)n'_1] + z_2[(1 + \pi)n'_2]$. In particular, \mathfrak{J} is additive and linear over E .

4.4. An algorithm for \mathfrak{J} . We now summarize the results of the previous section to give an explicit description of \mathfrak{J} when restricted to $\varphi(E \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathbb{Q}_p}^+)(1 + \pi)n'_1 \oplus \varphi(E \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathbb{Q}_p}^+)(1 + \pi)n'_2 \cong \varphi(E \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathbb{Q}_p}^+)^{\oplus 2}$. For a non-zero $y = (y_1, y_2) \in \varphi(E \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathbb{Q}_p}^+)^{\oplus 2}$, we write

$$y_1 = \sum_{n=0}^{\infty} a_n \varphi(\pi)^n \quad \text{and} \quad y_2 = \sum_{n=0}^{\infty} b_n \varphi(\pi)^n.$$

On multiplying by a power of ϖ_E , we may assume that $y \in \varphi(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)^{\oplus 2}$ but $\varpi_E \nmid y$. For such a y , we define the order $\text{ord}(y)$ of y to be the minimum integer n such that either a_n or b_n is a unit in \mathcal{O}_E .

Proposition 4.25. For y as above, there exists $z^{(n)} \in (\gamma - 1)^n \Lambda_{\mathcal{O}_E}(\Gamma)^2$ such that $y - \mathfrak{J}^{-1}(z^{(n)})$ has order strictly greater than n .

Proof. This is simply a reformulation of Proposition 4.22. In particular, one could take

$$z^{(n)} = (a_n(\gamma - 1)^n, b_n(\gamma - 1)^n).$$

\square

Corollary 4.26. For y as above, there exists a sequence $z^{(0)}, z^{(1)}, \dots$ in $\Lambda_{\mathcal{O}_E}(\Gamma)^{\oplus 2}$ such that $z^{(i)} \rightarrow 0$ as $i \rightarrow \infty$ and

$$y - \mathfrak{J}^{-1} \left(\sum_{i=0}^{\infty} z^{(i)} \right) \in \varpi_E \varphi(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)^2$$

We write $y^{(0)} = y$ and $u^{(0)}$ for the infinite sum given by Corollary 4.26. Define a sequence $y^{(n)}$ recursively: for $n \geq 0$, let $y^{(n+1)} = (y^{(n)} - \mathfrak{J}^{-1}(u^{(n)})) / \varpi_E$ where $u^{(n)}$ to be the sum given by Corollary 4.26 on applying it to $y^{(n)}$. Then, we have

$$\mathfrak{J}(y) = \sum_{i=0}^{\infty} \varpi_E^i u^{(i)}.$$

4.5. The image of $\underline{\text{Col}}_1$. Throughout this section, we assume that assumptions (B), (C) and (D) are satisfied.

Definition 4.27. Let $\underline{\text{Col}} = \mathfrak{J} \circ \text{Col} : \mathbb{N}(T)^{\psi=1} \rightarrow \Lambda_{\mathcal{O}_E}(G_\infty)^{\oplus 2}$, and for $i = 1, 2$, define

$$\underline{\text{Col}}_i : \mathbb{N}(T)^{\psi=1} \longrightarrow \Lambda_{\mathcal{O}_E}(G_\infty)$$

as the composition $\text{pr}_i \circ \underline{\text{Col}}$, where pr_i is the projection from $\Lambda_{\mathcal{O}_E}(G_\infty)^{\oplus 2}$ onto the i -th coefficient.

By abuse of notation, we also write $\underline{\text{Col}}$ for the natural extension $\mathbb{N}(V)^{\psi=1} \rightarrow \Lambda_E(G_\infty)$. The aim of this section is to prove the following theorem.

Theorem 4.28. The map $\underline{\text{Col}}_1 : \mathbb{N}(V)^{\psi=1} \rightarrow \Lambda_E(G_\infty)$ is surjective.

The idea of the proof is to translate Proposition 4.7 using the explicit description of \mathfrak{J} given in Section 4.4. Note that since \mathfrak{J} is a $\Lambda_E(G_\infty)$ -homomorphism, it is sufficient to show that $\varpi_E^m \in \text{Im}(\underline{\text{Col}}_1)$ for some $m \in \mathbb{Z}$.

Proposition 4.29. *Let $y_2 \in \varphi(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)$. Then there exists a sequence $z^{(i)} \in \Lambda_{\mathcal{O}_E}(\Gamma)$ tending to 0 as $i \rightarrow +\infty$ and $\tilde{x} \in \mathbb{N}(T)^{\psi=1}$ and $y'_2 \in \varphi(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)$ such that*

$$\mathfrak{J}(0, y_2) - \mathfrak{J} \circ \text{Col}(\tilde{x}) = \sum_{i \geq 0} (0, z^{(i)}) + \varpi_E \mathfrak{J}(0, y'_2).$$

Proof. If $(0, y_2) = +\infty$, then $\varpi_E|y_2$ and we are done. Assume that $\text{ord}(y_2) = n$ and write $y_2 = \sum_{r \geq 0} b_r \varphi(\pi)^r$. Then, by Lemma 4.25,

$$(47) \quad \mathfrak{J}(0, y_2) = \mathfrak{J}(y_1^{(1)}, y_2^{(1)}) + (0, b_n(\gamma - 1)^n).$$

where $y_2^{(i)}$ has order strictly greater than n . By applying \mathfrak{J}^{-1} to (47), we see that

$$y_2(1 + \pi)n'_2 = y_1^{(1)}(1 + \pi)n'_1 + y_2^{(1)}(1 + \pi)n'_2 + b_n(\gamma - 1)^n[(1 + \pi)n'_2].$$

Since G_γ is diagonal mod π^{k-1} , this implies that $y_1^{(1)} \equiv 0 \pmod{\varphi(\pi)^{k-1}}$. In particular, the proof of Proposition 4.6 implies that there exists $x_1 \in \mathbb{N}(T)^{\psi=1}$ such that $(1 - \varphi)x_1 = y_1^{(1)}(1 + \pi)n'_1$. Hence, we have

$$\mathfrak{J}(0, y_2) - \mathfrak{J} \circ \text{Col}(x_1) = \mathfrak{J}(0, y_2^{(1)}) + (0, z^{(1)})$$

where $z^{(1)} = b_n(\gamma - 1)^n$.

On applying the above to $y_2^{(1)}$ and repeat, we obtain sequences $\{x_n \in \mathbb{N}(T)^{\psi=1}\}$, $\{z^{(n)} \in \Lambda_{\mathcal{O}_E}(\Gamma)\}$ and $\{y_2^{(n)} \in \varphi(\mathbb{A}_{\mathbb{Q}_p}^+)\}$ such that

$$\mathfrak{J}(0, y_2^{(n-1)}) - \mathfrak{J} \circ \text{Col}(x_n) = \mathfrak{J}(0, y_2^{(n)}) + (0, z^{(n)}),$$

the sequence $m_n = \text{ord}(y_2^{(n)})$ is strictly increasing, $z^{(n)} \in (\gamma - 1)^{m_{n-1}} \Lambda_{\mathcal{O}_E}(\Gamma)$ and $\text{Col}(x_n) + (y_2^{(n)} - y_2^{(n-1)})(1 + \pi)n'_2 = z^{(n)}[(1 + \pi)n'_2]$. Now $\text{Col}(x_n) \in \varphi(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)(1 + \pi)n'_1$, so (i) $x_n \rightarrow 0$ and (ii) $(y_2^{(n)} - y_2^{(n-1)}) \rightarrow 0$.

By completeness, (ii) implies that $y_2^{(n)}$ converges to an element in $\varpi_E \varphi(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)$. (The limit must be in $\varpi_E \varphi(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)$ because the order of the limit is $+\infty$ by construction.) Now, on taking sums, we have for all $n \geq 1$,

$$\mathfrak{J}(0, y_2) - \sum_{i=1}^n \mathfrak{J} \circ \text{Col}(x_i) = \mathfrak{J}(0, y_2^{(n)}) + \sum_{i=1}^n (0, z^{(i)}).$$

We obtain the result by letting $n \rightarrow \infty$. \square

Corollary 4.30. *Let $y_2 \in \varphi(\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)$. Then there exists a sequence $z^{(i)} \in \Lambda_{\mathcal{O}_E}(\Gamma)$ tending to 0 as $i \rightarrow +\infty$ and $\tilde{x} \in \mathbb{N}(T)^{\psi=1}$ such that*

$$\mathfrak{J}(0, y_2) - \mathfrak{J} \circ \text{Col}(\tilde{x}) = \sum_{i \geq 0} (0, z^{(i)}).$$

Proof. Iterate the result in Proposition 4.29 for $\mathfrak{J}(0, y'_2)$ etc. and use that both $\Lambda_{\mathcal{O}_E}(\Gamma)$ and $\varphi^*(\mathbb{N}(T))^{\psi=0}$ are p -adically complete. \square

Corollary 4.31. *Let $y \in (\varphi^* \mathbb{N}(T))^{\psi=0}$ be of the form $y = y_2 n'_2$ for some $y_2 \in (\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$. Then there exists a sequence $z^{(i)} \in \Lambda_{\mathcal{O}_E}(G_\infty)$ tending to 0 as $i \rightarrow +\infty$ and $\tilde{x} \in \mathbb{N}(T)^{\psi=1}$ such that*

$$\mathfrak{J}(y) - \mathfrak{J} \circ \text{Col}(\tilde{x}) = \sum_{i \geq 0} (0, z^{(i)}).$$

Proof. Immediate from the previous corollary and the observation that $(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0} n'_2$ is the Δ -orbit of $\varphi(\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)(1 + \pi)n'_2$. \square

We can now prove Theorem 4.28. By Proposition 4.7 there exists $x \in \mathbb{N}(T)^{\psi=1}$ such that $\text{Col}(x) = \varpi_E^m(1 + \pi)n'_1 + y_2n'_2$ for some $y_2 \in (\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$. It is clear that $\mathfrak{J}(\varpi_E^m(1 + \pi)n'_1) = (\varpi_E^m, 0)$. Also, we know by Corollary 4.31 that there exists a sequence $z^{(i)} \in \Lambda_{\mathcal{O}_E}(G_\infty)$ tending to 0 as $i \rightarrow +\infty$ and $\tilde{x} \in \mathbb{N}(T)^{\psi=1}$ such that

$$\mathfrak{J}(y_2n'_2) - \mathfrak{J} \circ \text{Col}(\tilde{x}) = \sum_{i \geq 0} (0, z^{(i)}).$$

Hence $\mathfrak{J} \circ \text{Col}(x) - \mathfrak{J} \circ \text{Col}(\tilde{x}) = (\varpi_E^m, 0) + \sum_{i \geq 0} (0, z^{(i)})$, i.e.

$$\mathfrak{J} \circ \text{Col}(x - \tilde{x}) = (\varpi_E^m, 0) + \sum_{i \geq 0} (0, z^{(i)}).$$

Remark 4.32. *Alas so far we don't know how to translate Proposition 4.11 into a statement about $\text{Im } \underline{\text{Col}}_2$.*

Remark 4.33. *In a forthcoming paper [LLZ10], we give a description of the images of the $\underline{\text{Col}}_i$ using Perrin-Riou's p -adic regulator.*

5. RELATIONS TO EXISTING WORK

5.1. Fourier transforms. In this section, we prove a compatibility result in p -adic Fourier theory (theorem 5.4 below) which will allow us to relate divisibility of elements in $\mathcal{H}(G_\infty)$ and of their images in $(\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0}$ under the Mellin transform. This will allow us to compare our results above to the ones in [Kob03], [Lei09] and [Spr09]. Throughout, E is a complete extension of \mathbb{Q}_p .

5.1.1. The Fourier transform for \mathbb{Z}_p and \mathbb{Z}_p^\times . We recall some standard results of p -adic Fourier theory. These results are due to Amice [AV75]; see also [Col10] for a more modern account. We denote by $C^{\text{la}}(\mathbb{Z}_p, E)$ the space of locally analytic E -valued functions on \mathbb{Z}_p , with the topology it acquires as the locally convex direct limit as $n \rightarrow \infty$ of the Banach algebras of functions analytic on cosets of $p^n\mathbb{Z}_p$. A *distribution* on \mathbb{Z}_p is a continuous E -linear functional $C^{\text{la}}(\mathbb{Z}_p, E) \rightarrow E$; we write $D^{\text{la}}(\mathbb{Z}_p, E)$ for the space of distributions.

Proposition 5.1 ([Col10, theorem 2.3]). *There is an isomorphism between $D^{\text{la}}(\mathbb{Z}_p, E)$ and the subset of functions $f \in E[[T]]$ converging for all T in the open unit disc of \mathbb{C}_p , given by $\mu \mapsto F_\mu(T) = \sum_{n \geq 0} T^n \mu \left(\binom{x}{n} \right)$. The value of F_μ at a point $x \in E$ (with $|x| < 1$) is $\mu(\kappa_x)$, where κ_x is the unique character of \mathbb{Z}_p such that $\kappa(1) = 1 + x$.*

Thus we may identify $D^{\text{la}}(\mathbb{Z}_p, E)$ with $E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+$. Under this identification, the subspace $D^{\text{la}}(\mathbb{Z}_p^\times, E)$ of distributions supported in \mathbb{Z}_p^\times corresponds to $(E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0}$ [Col10, §2.4.5].

Suppose $p \neq 2$. An alternative description of $D^{\text{la}}(\mathbb{Z}_p^\times, E)$ is given by the isomorphism $\mathbb{Z}_p^\times = (1 + p\mathbb{Z}_p) \times \Delta \cong \mathbb{Z}_p \times \Delta$, where Δ is the group of $(p-1)$ st roots of unity in \mathbb{Z}_p . If we fix a topological generator γ of $1 + p\mathbb{Z}_p$, we thus have an isomorphism

$$D^{\text{la}}(\mathbb{Z}_p^\times, E) \cong E \otimes \mathcal{H}(G_\infty),$$

where as in section 3.4 above, $\mathcal{H}(G_\infty)$ is the ring of formal series $f(\gamma - 1)$, for $f \in \mathbb{Q}_p[\Delta][[X]]$ converging for all $|X| < 1$.

Thus for a distribution μ on \mathbb{Z}_p^\times , we obtain two power series

$$F_\mu^+(\pi) \in (E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0}$$

and

$$F_\mu^\times(X) \in E \otimes \mathcal{H}(G_\infty).$$

These are related by the Mellin transform of lemma 3.7: we have $\mathfrak{M}(F_\mu^\times(\gamma)) = F_\mu^+$.

5.1.2. *Step functions.* Let $n \geq 0$ be an integer. We say a function $f : \mathbb{Z}_p \rightarrow E$ is a *step function of order n* if it is constant on any coset $a + p^n \mathbb{Z}_p$; the space $\text{Step}_n(\mathbb{Z}_p)$ of such functions is clearly a subspace of $C^{\text{la}}(\mathbb{Z}_p, E)$ of dimension p^n .

For each n we have an inclusion $\text{Step}_n(\mathbb{Z}_p) \rightarrow \text{Step}_{n+1}(\mathbb{Z}_p)$. A section of this is given by the “averaging” map $I : \text{Step}_{n+1}(\mathbb{Z}_p) \rightarrow \text{Step}_n(\mathbb{Z}_p)$ defined by

$$I(f)(x) = \frac{1}{p} \sum_{y \in \mathbb{Z}/p\mathbb{Z}} f(x + p^n y).$$

For $n \geq 1$, we say a function $f \in \text{Step}_n(\mathbb{Z}_p)$ is a *primitive step function* if it is in the kernel of this map, and write $\text{PStep}_n(\mathbb{Z}_p)$ for the space of such functions, which clearly has dimension $p^{n-1}(p-1)$. For consistency we take $\text{PStep}_0(\mathbb{Z}_p) = \text{Step}_0(\mathbb{Z}_p) = K$.

Lemma 5.2. *Let $n \geq 0$ and suppose E contains a primitive p^n -th root of unity ζ_{p^n} . Then a basis for $\text{Step}_n(\mathbb{Z}_p)$ is given by the functions $x \mapsto (\zeta_{p^n})^{xt}$, as t varies through $\mathbb{Z}/p^n \mathbb{Z}$. The subset corresponding to $t \in (\mathbb{Z}/p^n \mathbb{Z})^\times$ is a basis for $\text{PStep}_n(\mathbb{Z}_p)$.*

Proof. This follows immediately from the fact that $x \mapsto \frac{1}{p^n} \sum_{t \in \mathbb{Z}/p^n \mathbb{Z}} (\zeta_{p^n})^{xt} (\zeta_{p^n})^{-at}$ is the characteristic function of $a + p^n \mathbb{Z}_p$. \square

We also have a “multiplicative” version. For $n \geq 1$, we define $\text{Step}_n(\mathbb{Z}_p^\times)$ as the functions in $\text{Step}_n(\mathbb{Z}_p)$ which are supported in \mathbb{Z}_p^\times . For $n \geq 2$ the averaging map restricts to a map $\text{Step}_n(\mathbb{Z}_p^\times) \rightarrow \text{Step}_n(\mathbb{Z}_p^\times)$, and we define $\text{PStep}_n(\mathbb{Z}_p^\times)$ to be its kernel. We take $\text{PStep}_1(\mathbb{Z}_p^\times) = \text{Step}_1(\mathbb{Z}_p^\times)$, so for all $n \geq 1$ restriction to \mathbb{Z}_p^\times defines a surjective map $\text{PStep}_n(\mathbb{Z}_p) \rightarrow \text{PStep}_n(\mathbb{Z}_p^\times)$.

Lemma 5.3. *A basis for $\text{Step}_n(\mathbb{Z}_p^\times)$ is given by the Dirichlet characters modulo p^n . For $n \geq 2$ the subset of primitive characters modulo p^n gives a basis for $\text{PStep}_n(\mathbb{Z}_p^\times)$.*

Proof. Similar to the previous lemma. \square

5.1.3. *Relating the additive and multiplicative transforms.* We now suppose we are given a distribution $\mu \in D^{\text{la}}(\mathbb{Z}_p^\times, E)$. Let F_μ^\times and F_μ^+ be the corresponding transforms.

Theorem 5.4. *For $n \geq 2$, the following are equivalent:*

- (1) F_μ^+ is divisible by the cyclotomic polynomial $\Phi_n(1 + \pi)$ in $\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+$.
- (2) μ annihilates $\text{PStep}_n(\mathbb{Z}_p)$.
- (3) μ annihilates $\text{PStep}_n(\mathbb{Z}_p^\times)$.
- (4) $F_\mu^\times(\chi)$ is zero for all primitive Dirichlet characters $\chi \bmod p^n$.
- (5) F_μ^\times is divisible by $\Phi_{n-1}(1 + X)$ in $E \otimes \mathcal{H}(G_\infty)$.

For $n = 1$, the same holds with the last two statements replaced by:

- (4') $F_\mu^\times(\chi)$ is zero for all Dirichlet characters $\chi \bmod p$.
- (5') F_μ^\times is divisible by X in $E \otimes \mathcal{H}(G_\infty)$.

Proof. It is clear that (1) \Leftrightarrow (2) for arbitrary $\mu \in D^{\text{la}}(\mathbb{Z}_p, E)$ (not necessarily supported in \mathbb{Z}_p^\times), because of Lemma 5.2. Since restriction of functions gives a surjective map $\text{PStep}_n(\mathbb{Z}_p) \rightarrow \text{PStep}_n(\mathbb{Z}_p^\times)$, we have (2) \Leftrightarrow (3). The equivalence (3) \Leftrightarrow (4) follows from Lemma 5.3.

To show (4) \Leftrightarrow (5) for $n \geq 2$, let us write $F_\mu^\times = \sum_{i=1}^{p-1} [\tau(i)] F_i(X)$, where $F_i \in E[[X]]$ and $\tau(i) \in \Delta$ is the Teichmüller lift of i . For any primitive p^{n-1} st root of unity ζ , there are exactly $p-1$ primitive Dirichlet characters modulo p^n mapping γ to ζ , and their restrictions to Δ are given by $\tau(i) \mapsto \tau(i)^k$ for $k \in \mathbb{Z}/(p-1)\mathbb{Z}$. So (4) is equivalent to

$$\sum_{i=1}^{p-1} \tau(i)^k F_i(\zeta - 1) = 0$$

for all $k = 0 \dots p-2$ and all primitive p^{n-1} st roots of unity ζ , which is equivalent to $F_i(\zeta - 1) = 0$ for each $i = 1, \dots, p-1$. In other words, each of the functions $F_i(X)$ vanishes at every root of the polynomial $\Phi_{n-1}(1+X)$, which is clearly equivalent to F_μ^\times being divisible by $\Phi_{n-1}(1+X)$ in $E \otimes \mathcal{H}(G_\infty)$.

(The only change necessary for $n = 1$ is to note that $\text{PStep}_1(\mathbb{Z}_p^\times)$ is the linear span of all Dirichlet characters modulo p , not just the primitive ones.) \square

We also have an accompanying result:

Lemma 5.5. *Let $F \in (E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0}$. Then $\Phi_1(1+\pi)$ divides $F(\pi)$ if and only if $\varphi(\pi) = \pi\Phi_1(1+\pi)$ divides $F(\pi)$.*

Proof. Since $\psi(F)(0) = 0$, we have

$$F(0) + \sum_{\substack{\zeta \in \mu_p \\ \zeta \neq 1}} F(\zeta - 1) = 0.$$

Hence if F vanishes at the points $\zeta - 1$ for primitive $\zeta \in \mu_p$, then it must also vanish at 0. \square

5.2. The case $a_p = 0$. We now relate the construction of Coleman maps in this paper to the construction given in [Lei09] for modular forms with $a_p = 0$.

5.2.1. Construction of the Coleman maps. Consider f a normalized new eigenform as in Section 3.3 with $a_p = 0$. To ease notation, we assume that $E = \mathbb{Q}_p$. The plus and minus Coleman maps in [Lei09] are constructed as follows.

Let $u = \chi(\gamma)$. In [Pol03], Pollack defines the following elements of $\mathcal{H}(G_\infty)$:

$$\begin{aligned} \log_{p,k}^+ &= \prod_{j=0}^{k-2} \frac{1}{p} \prod_{n=1}^{+\infty} \frac{\Phi_{2n}(u^{-j}\gamma)}{p} \\ \log_{p,k}^- &= \prod_{j=0}^{k-2} \frac{1}{p} \prod_{n=1}^{+\infty} \frac{\Phi_{2n-1}(u^{-j}\gamma)}{p}. \end{aligned}$$

Let $\nu^- = \bar{\nu}_1$, $\nu^+ = \bar{\nu}_2$ be the basis of $\mathbb{D}_{\text{cris}}(V_{\bar{f}})$ as in Section 3.3 and let $\eta^\pm = (1+\pi) \otimes \nu^\pm \in (\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)^{\psi=0} \otimes \mathbb{D}_{\text{cris}}(V)$. Let

$$\mathcal{L}_{1,\eta^\pm} : H_{\text{Iw}}^1(\mathbb{Q}_p, V_{\bar{f}}(k-1)) \longrightarrow \mathcal{H}(G_\infty)$$

be the map defined by (18).

Lemma 5.6. $\log_{p,k}^\pm \mid \mathcal{L}_{1,\eta^\pm}(z)$ for any $z \in H_{\text{Iw}}^1(\mathbb{Q}_p, V_{\bar{f}}(k-1))$.

Proof. See [Lei09, Lemma 2.2]. \square

One can therefore define

$$\begin{aligned} (48) \quad \text{Col}^\pm : H_{\text{Iw}}^1(\mathbb{Q}_p, V_{\bar{f}}(k-1)) &\longrightarrow \Lambda_{\mathbb{Q}_p}(G_\infty) \\ z &\longmapsto \frac{\mathcal{L}_{1,\eta^\pm}(z)}{\log_{p,k}^\pm}. \end{aligned}$$

In this setting, we can work out the matrix M in (20) explicitly. As in section 4 above, we let n_1, n_2 be the basis of $\mathbb{N}(V_{\bar{f}})$ constructed in [BLZ04]. The results of *op.cit.* imply that the $\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+$ -span of n_1, n_2 is $\mathbb{N}(T_{\bar{f}})$ for a $G_{\mathbb{Q}_p}$ -stable \mathcal{O}_E -lattice $T_{\bar{f}} \subset V_{\bar{f}}$.

Recall that $M \in M_2(\varphi(\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+))$ is the matrix satisfying $\begin{pmatrix} \varphi(n_1) \\ \varphi(n_2) \end{pmatrix} = M \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$.

Lemma 5.7. *The matrix M is given by*

$$\begin{pmatrix} 0 & (\log^+(1+\pi))^{k-1} \\ -(\log^-(1+\pi)/q)^{k-1} & 0 \end{pmatrix}.$$

Proof. With respect to the basis n_1, n_2 of $\mathbb{N}(V_{\bar{f}})$ over $\mathbb{B}_{\mathbb{Q}_p}^+$, as chosen in [BLZ04], the matrices of φ and $\gamma \in G_\infty$ are given by

$$(49) \quad P = \begin{pmatrix} 0 & -1 \\ q^{k-1} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \left(\frac{\log^+(1+\pi)}{\gamma(\log^+(1+\pi))}\right)^{k-1} & 0 \\ 0 & \left(\frac{\log^-(1+\pi)}{\gamma(\log^-(1+\pi))}\right)^{k-1} \end{pmatrix}$$

respectively. Then,

$$\bar{\nu}_1 = (\log^+(1+\pi))^{k-1} n_1 \quad \text{and} \quad \bar{\nu}_2 = (\log^-(1+\pi))^{k-1} n_2,$$

so the base-change matrix M' (defined in (8)) is given by

$$(50) \quad \begin{pmatrix} (\log^+(1+\pi))^{k-1} & 0 \\ 0 & (\log^-(1+\pi))^{k-1} \end{pmatrix}$$

and the result follows from explicit calculations, using that $M = \left(\frac{t}{\pi q}\right)^{k-1} P^T M'^{-1}$. \square

Lemma 5.8. *We have $\varphi(\log^-(1+\pi)) = \log^+(1+\pi)$ and $\varphi(\log^+(1+\pi)) = \frac{p}{q} \log^-(1+\pi)$.*

Proof. Immediate. \square

Lemma 5.9. *For $i \in \{1, \dots, p-1\}$ we have*

$$\mathfrak{M}^{-1} \left((1+\pi)^i \log^+(1+\pi)^{k-1} \cdot \varphi(\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+) \right) = \tau(i) \log_{p,k}^-(\gamma) \cdot \mathcal{H}(\Gamma)$$

and

$$\mathfrak{M}^{-1} \left((1+\pi)^i \log^-(1+\pi)^{k-1} / q^{k-1} \cdot \varphi(\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+) \right) = \tau(i) \log_{p,k}^+(\gamma) \cdot \mathcal{H}(\Gamma),$$

where $\tau(i) \in \Delta$ is the Teichmuller lift of i .

Proof. Let us suppose first that $k = 2$. Any element $f \in (1+\pi)^i \log^+(1+\pi) \cdot \varphi(\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)$ is F_μ^+ for some distribution μ on \mathbb{Z}_p , supported in $i + p\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$; hence we have a corresponding multiplicative Fourier transform $F_\mu^\times = \mathfrak{M}^{-1}(f)$, lying in $\tau(i)\mathcal{H}(\Gamma)$. Moreover, we have the implications

$$\begin{aligned} & \log^+(1+\pi) \mid f \text{ in } \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+ \\ \iff & \Phi_n(1+\pi) \mid f \text{ for all even } n \geq 2 \\ \iff & \Phi_n(1+X) \mid \mathfrak{M}^{-1}(f) \text{ for all odd } n \geq 1 \text{ (by theorem 5.4)} \\ \iff & \log^-(1+X) \mid \mathfrak{M}^{-1}(f). \end{aligned}$$

The second statement is similar, noting that $q = \Phi_1(1+\pi)$ and hence $\log^-(1+\pi)/q$ divides f if and only if f vanishes at the primitive p^n -th roots of unity for all odd $n \geq 3$.

For general $k \geq 2$, we note that $f \in \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+$ vanishes to order $k-1$ at a point z if and only if $\partial^j f$ vanishes at z for $j = 0, \dots, k-2$, where ∂ is the differential operator $(1+\pi) \frac{d}{d\pi}$ introduced in §2.2. Applying the preceding argument to each of the functions $\partial^j f$, we see that $\log^+(1+\pi) \mid f$ if and only if $\mathfrak{M}^{-1}(\partial^j f)$ is divisible by $\log^-(1+X)$ for $0 \leq j \leq k-2$. Since $\mathfrak{M}^{-1}(\partial^j f)(z) = \mathfrak{M}^{-1}(f)(u^j(1+z) - 1)$ where $u = \chi(\gamma)$, this is equivalent to the divisibility of f by $\log_{p,k}^+$. Again, the second statement follows very similarly to the first. \square

Proposition 5.10. *There exists $a^\pm \in \Lambda_E(G_\infty)^\times$ such that*

$$\underline{M} = \begin{pmatrix} 0 & -a^- \log_{p,k}^- \\ a^+ \log_{p,k}^+ & 0 \end{pmatrix}.$$

Proof. By Lemma 5.9, \mathfrak{M} restricts to an isomorphism of $\mathcal{H}(G_\infty)$ -modules between the subspaces $X^\pm = (1 + \pi)\varphi(\log^\pm(1 + \pi))^{k-1} \cdot \varphi(\mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+)$ and $Y^\pm = \log_{p,k}^\pm \cdot \mathcal{H}(G_\infty)$. In particular, there exist $a^\pm \in \mathcal{H}(G_\infty)$ such that

$$\mathfrak{M}^{-1}((1 + \pi)\varphi(\log^\pm(1 + \pi))^{k-1}) = a^\pm \log_{p,k}^\pm.$$

Furthermore, $(1 + \pi)\varphi(\log^\pm(1 + \pi))^{k-1}$ are $\Lambda_E(G_\infty)$ -module generators of $(1 + \pi)\varphi(\log^\pm(1 + \pi))^{k-1} \cdot \varphi(\mathbb{B}_{\mathbb{Q}_p}^+)$, by Proposition 4.24. Since any finitely-generated submodule of $\mathcal{H}(G_\infty)$ is closed, they must be $\mathcal{H}(G_\infty)$ -module generators of the closures of these spaces, which are clearly X^\pm . Therefore the images of $(1 + \pi)\varphi(\log^\pm(1 + \pi))^{k-1}$ under \mathfrak{M}^{-1} must be generators of Y^\pm , so the factors a^\pm are units. \square

Therefore, by (32), we have:

Corollary 5.11. *Let a^\pm be as in Proposition 5.10, then $a^- \underline{\text{Col}}_1 = \text{Col}^-$ and $a^+ \underline{\text{Col}}_2 = \text{Col}^+$.*

5.2.2. Description of the kernels. The aim of this section is to give a simple description of $\ker(\text{Col}_i)$ for $i = 1, 2$. Recall that the basis $\bar{\nu}_1, \bar{\nu}_2$ of $\mathbb{D}_{\text{cris}}(V_{\bar{f}})$ determines a basis of $\mathbb{D}_{\text{cris}}(V_{\bar{f}}(k-1))$ via the map $\bar{\nu}_i \mapsto \bar{\nu}_i \otimes e_{k-1}t^{1-k}$. We first need to know a bit more about $\mathbb{N}(V_{\bar{f}})$. As stated in [Ber03, Section II.3], we have a comparison isomorphism

$$\iota : \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+[t^{-1}] \otimes_{\mathbb{B}_{\mathbb{Q}_p}^+} \mathbb{N}(V_{\bar{f}}(k-1)) \cong \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+[t^{-1}] \otimes_{\mathbb{Q}_p} \mathbb{D}_{\text{cris}}(V_{\bar{f}}(k-1)).$$

By (20) and (10), if $x \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$, then we can write $\iota(x) = x_1(\bar{\nu}_1 \otimes e_{k-1}t^{1-k}) + x_2(\bar{\nu}_2 \otimes e_{k-1}t^{1-k})$ where

$$\begin{aligned} x_1 &= x'_1(\log^-(1 + \pi))^{k-1} \\ x_2 &= x'_2(\log^+(1 + \pi))^{k-1} \end{aligned}$$

for some $x'_1, x'_2 \in \mathbb{B}_{\mathbb{Q}_p}^+$.

We will need the following auxiliary lemma.

Lemma 5.12. *Let x be as above. Then $p^{k-2}\theta(x_1) + \theta(x_2) = 0$.*

Proof. By [Ber03, Theorem II.6], we have

$$(51) \quad \exp_{\mathbb{Q}_p, V_{\bar{f}}(k-1)}^*(h_{\mathbb{Q}_p, V_{\bar{f}}(k-1)}^1(x)) = (1 - p^{-1}\varphi^{-1})\partial_V(x).$$

Since $\partial_V(x) = \theta(x_1)\bar{\nu}_1 \otimes e_{k-1}t^{1-k} + \theta(x_2)\bar{\nu}_2 \otimes e_{k-1}t^{1-k}$, we have

$$(1 - p^{-1}\varphi^{-1})\partial_V(x) = (\theta(x_1) - p^{-1}\theta(x_2))\nu_1 + (p^{k-2}\theta(x_1) + \theta(x_2))\nu_2.$$

The image of $\exp_{\mathbb{Q}_p, V_{\bar{f}}(k-1)}^*$ is contained in $\text{Fil}^0 \mathbb{D}_{\text{cris}}(V_{\bar{f}}(k-1))$, which implies that $p^{k-2}\theta(x_1) + \theta(x_2) = 0$. \square

Lemma 5.13. *Let $x \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$, and write $\iota(x) = x_1(\bar{\nu}_1 \otimes e_{k-1}t^{1-k}) + x_2(\bar{\nu}_2 \otimes e_{k-1}t^{1-k})$ as above. Then*

- (i) $x \in \ker(\text{Col}_1)$ if and only if $\varphi(x_1) = -p^{k-1}\psi(x_1)$;
- (ii) $x \in \ker(\text{Col}_2)$ if and only if $\varphi(x_2) = -p^{k-1}\psi(x_2)$.

Proof. We will prove the proposition for Col_1 ; the proof for Col_2 is analogous. Note that the condition that $\psi(x) = x$ translates as $\psi(x_1) = -p^{1-k}x_2$ and $\psi(x_2) = x_1$. By Lemma 5.8, $\text{Col}_1(x) = x'_2 - \varphi(x'_1) = 0$ if and only if $x_2 = \varphi(x_1)$. Hence, $\text{Col}_1(x) = 0$ if and only if $\varphi(x_1) = -p^{k-1}\psi(x_1)$. \square

Proposition 5.14. *Let x be as above, and write $x_i = f_i(\pi)$ with $f_i(X) \in \mathbb{Q}_p[[X]]$. Then*

- (i) $x \in \ker(\text{Col}_1)$ if and only if

$$(52) \quad \text{Tr}_{\mathbb{Q}_{p,n}/\mathbb{Q}_{p,n-1}}(f_1(\zeta_{p^n} - 1)) = -p^{2-k}f_1(\zeta_{p^{n-2}} - 1) \text{ for all } n \geq 2, \text{ and}$$

$$(53) \quad \text{Tr}_{\mathbb{Q}_{p,1}/\mathbb{Q}_p}(f_1(\zeta_p - 1)) = -(1 + p^{2-k})f_1(0);$$

(ii) $x \in \ker(\text{Col}_2)$ if and only if

$$\begin{aligned} \text{Tr}_{\mathbb{Q}_{p,n}/\mathbb{Q}_{p,n-1}}(f_2(\zeta_{p^n} - 1)) &= -p^{2-k} f_2(\zeta_{p^{n-2}} - 1) \text{ for all } n \geq 2, \text{ and} \\ \text{Tr}_{\mathbb{Q}_{p,1}/\mathbb{Q}_p}(f_2(\zeta_p - 1)) &= -(1 + p^{2-k}) f_2(0). \end{aligned}$$

Proof. We prove the proposition for Col_1 . Recall that

$$\varphi\psi(x_1) = p^{-1} \sum_{\zeta^p=1} f_1(\zeta(1+\pi) - 1).$$

Hence, $\varphi(x_1) = -p^{k-1}\psi(x_1)$ implies that

$$(54) \quad \sum_{\zeta^p=1} f_1(\zeta(1+\pi) - 1) = -p^{2-k} \varphi^2(f_1(\pi)).$$

Let $n \geq 2$. On applying $\theta \circ \varphi^{-n}$ to (54) implies that

$$\text{Tr}_{\mathbb{Q}_{p,n}/\mathbb{Q}_{p,n-1}}(f_1(\zeta_{p^n} - 1)) = \sum_{\zeta^p=1} f_1(\zeta\zeta_{p^n} - 1) = -p^{2-k} f_1(\zeta_{p^{n-2}} - 1).$$

Similarly, we obtain the second condition by applying θ to (54).

Conversely, assume that (52) holds for all $n \geq 2$, then $\varphi(f_1) + p^{k-1}\psi(f_1) = 0$ at $\zeta_{p^n} - 1$. Recall that $x_1 = x'_1(\log^-(1+\pi))^{k-1}$ where $x'_1 \in \mathbb{B}_{\mathbb{Q}_p}^+$. By Lemma 5.8,

$$\varphi(x_1) + p^{k-1}\psi(x_1) = (\varphi(x'_1) + \psi(q^{k-1}x'_1))(\log^+(1+\pi))^{k-1}.$$

Hence, the power series in $\mathbb{Q} \otimes \mathbb{Z}_p[[X]]$ corresponding to $(\varphi(x'_1) + \psi(q^{k-1}x'_1))$ has infinitely many zeros, so it must be zero itself and we are done. \square

As a corollary, we obtain the following descriptions of $\ker(\text{Col}_i)$.

Corollary 5.15. *For $x \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1}$, write $e_n(x) = \exp_{n, V_{\bar{f}}(k-1)}^* \circ \text{Pr}_n \circ h_{\mathbb{Q}_p, \text{Iw}}^1(x)$ where Pr_n is the projection from $H_{\text{Iw}}^1(\mathbb{Q}_p, V_{\bar{f}}(k-1))$ to $H^1(\mathbb{Q}_{p,n}, V_{\bar{f}}(k-1))$. Let $i = 1$ (respectively $i = 2$), then*

$$\ker(\text{Col}_i) = \{x \in \mathbb{D}(V_{\bar{f}}(k-1))^{\psi=1} : e_0(x) = 0 \text{ and } e_{n+1}(x) = p^{-1}e_n(x) \forall \text{ odd (respectively even) } n \geq 1\}.$$

Proof. Again, we only prove this for $i = 1$. By [CC99, Théorème IV.2.1], we have $e_n(x) = p^{-n}\partial_V(\varphi^{-n}(x))$ for all $n \geq 1$. But φ^{-2} is the multiplication by $-p^{k-1}$ on $\mathbb{D}_{\text{cris}}(V_{\bar{f}}(k-1))$. Using again that $\text{Im}(\exp_{n, V_{\bar{f}}(k-1)}^*) \subset \text{Fil}^0 \mathbb{D}_{\text{cris}}(V)$, we see that

$$\begin{aligned} e_{2n}(x) &= p^{-2n} \cdot (-p)^{n(k-1)} f_1(\zeta_{p^{2n}} - 1) \bar{\nu}_1 \otimes t^{1-k} e_{k-1} \\ e_{2n+1}(x) &= p^{-2n-1} \cdot (-p)^{n(k-1)} f_2(\zeta_{p^{2n+1}} - 1) \bar{\nu}_1 \otimes t^{1-k} e_{k-1} \end{aligned}$$

and $f_2(\zeta_{p^{2n}} - 1) = f_1(\zeta_{p^{2n-1}} - 1) = 0$ for all $n \geq 1$. Therefore, (52) holds for $2n-1$ and for $2n$ if and only if $e_{2n}(x) = \text{Tr}_{F_{2n+1}/F_{2n}}(e_{2n+1}(x)) = p^{-1}e_{2n-1}(x)$.

Now $e_0(x) = (f_1(0) - p^{-1}f_2(0))\bar{\nu}_1 \otimes t^{1-k} e_{k-1}$ by (51) and $p^{k-2}f_1(0) + f_2(0) = 0$ by Lemma 5.12, so

$$e_0(x) = (1 + p^{k-3})f_1(0)\bar{\nu}_1 \otimes t^{1-k} e_{k-1} = -(p^{2-k} + p^{-1})f_2(0)\bar{\nu}_1 \otimes t^{1-k} e_{k-1}$$

The condition (53) is therefore equivalent to $f_1(0) = 0$, which in turns is equivalent to $e_0(x) = 0$. \square

In the rest of this section, we will relate Corollary 5.15 to the description of $\ker(\text{Col}^\pm)$ in [Lei09, Section 2.2]. Recall that $H_f^1(\mathbb{Q}_{p,n}, T_f(1))^\pm$ is defined by

$$\{x \in H_f^1(\mathbb{Q}_{p,n}, T_f(1)) : \text{cor}_{n/m+1}x \in H_f^1(\mathbb{Q}_{p,m}, T_f(1)) \forall m \text{ even (odd), } m < n\}.$$

Denote by $H_\pm^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(k-1))$ the annihilator of $H_f^1(\mathbb{Q}_{p,n}, T_f(1))^\pm$ under the pairing

$$(55) \quad [,]_n : H^1(\mathbb{Q}_{p,n}, T_f(1)) \times H^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(k-1)) \rightarrow \mathbb{Z}_p.$$

As shown in [Lei09, Section 2.2.4], we have $\ker(\text{Col}^\pm) = \varprojlim_n H_\pm^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(k-1))$. Hence, we can identify the kernels described in Corollary 5.15 with $\ker(\text{Col}^\pm)$ described in [Lei09] via the isomorphism $h_{\text{Iw}, V_{\bar{f}}(k-1)}^1$:

Proposition 5.16. *For any $x \in H^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(k-1))$ and $m \leq n$, let $e_m(x) = \exp_{m, V_{\bar{f}}(k-1)}^*(\text{cor}_{n/m}(x))$. Then, $H_{\pm}^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(k-1))$ coincides with the following set:*

$$\{x \in H^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(k-1)) : e_0(x) = 0 \text{ and } e_m(x) = p^{-1}e_{m-1}(x) \forall m \text{ odd (even), } m \leq n\}$$

Proof. On the one hand, (55) factors through

$$H_{\bar{f}}^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(1)) \times \frac{H^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(k-1))}{H_{\bar{f}}^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(k-1))} \rightarrow \mathbb{Z}_p.$$

On the other hand, the pairing

$$[\sim, \sim]_n' : \left(\mathbb{Q}_{p,n} \otimes \mathbb{D}_{\text{cris}}(V_{\bar{f}}(1)) \right) \times \left(\mathbb{Q}_{p,n} \otimes \mathbb{D}_{\text{cris}}(V_{\bar{f}}(k-1)) \right) \rightarrow \mathbb{Q}_{p,n} \xrightarrow{\text{Tr}_{n/0}} \mathbb{Q}_p$$

factors through

$$\left(\mathbb{Q}_{p,n} \otimes \mathbb{D}_{\text{cris}}(V_{\bar{f}}(1)) / \mathbb{D}_{\text{cris}}^0(V_{\bar{f}}(1)) \right) \times \left(\mathbb{Q}_{p,n} \otimes \mathbb{D}_{\text{cris}}^0(V_{\bar{f}}(k-1)) \right) \rightarrow \mathbb{Q}_p.$$

Hence, the compatibility of the two pairings, namely $[\exp_{n, V_{\bar{f}}(1)}(\sim), \sim]_n = \text{Tr}_{n/0}[\sim, \exp_{n, V_{\bar{f}}(k-1)}^*(\sim)]_n'$, implies that $H_{\pm}^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(k-1))$ is the $\exp_{n, V_{\bar{f}}(k-1)}^*$ -preimage of $\left(\mathbb{Q}_{p,n}^{\pm} \otimes \mathbb{D}_{\text{cris}}(V_{\bar{f}}(1)) / \mathbb{D}_{\text{cris}}^0(V_{\bar{f}}(1)) \right)^{\perp}$ where

$$\mathbb{Q}_{p,n}^{\pm} = \{x \in \mathbb{Q}_{p,n} : \text{Tr}_{n/m+1}(x) \in \mathbb{Q}_{p,m} \forall m \text{ even (odd), } m < n\}.$$

But we have:

$$\left(\mathbb{Q}_{p,n}^{\pm} \otimes \mathbb{D}_{\text{cris}}(V_{\bar{f}}(1)) / \mathbb{D}_{\text{cris}}^0(V_{\bar{f}}(1)) \right)^{\perp} = \left(\mathbb{Q}_{p,n}^{\pm} \right)^{\perp} \otimes \mathbb{D}_{\text{cris}}^0(V_{\bar{f}}(k-1))$$

where $\left(\mathbb{Q}_{p,n}^{\pm} \right)^{\perp}$ is the orthogonal complement of $\mathbb{Q}_{p,n}^{\pm}$ under the pairing

$$\begin{aligned} \mathbb{Q}_{p,n} \times \mathbb{Q}_{p,n} &\rightarrow \mathbb{Q}_p \\ (x, y) &\mapsto \text{Tr}_{n/0}(xy). \end{aligned}$$

By simple linear algebra, we have

$$\left(\mathbb{Q}_{p,n}^{\pm} \right)^{\perp} = \{x \in \mathbb{Q}_{p,n} : \text{Tr}_{n/0}(x) = 0 \text{ and } \text{Tr}_{n/m+1}(x) \in \mathbb{Q}_{p,m} \forall m \text{ odd (even), } m < n\},$$

hence the lemma. \square

5.3. Elliptic curves with $a_p = 0$. We now specialize to the case when f corresponds to an elliptic curve E over \mathbb{Q} with $a_p = 0$. Then $V_{\bar{f}}(k-1) = V_{\bar{f}}(1) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$, where $T = T_p(E)$. Furthermore, $E[p]$ is irreducible as a mod p representation of $G_{\mathbb{Q}_p}$; thus T is the unique $G_{\mathbb{Q}_p}$ -stable lattice in $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(E)$ up to scaling, and in particular we may take the lattice $T_{\bar{f}}(1)$ constructed in [BLZ04] (which is only defined up to scaling) to coincide with T .

In this situation, we can recover results of Kobayashi [Kob03] which give a precise description of the images $\mathbb{D}(T)^{\psi=1}$ under the Coleman maps. Recall that if $x \in \mathbb{D}(V)^{\psi=1}$, say $x = (x_1 n_1 + x_2 n_2) \otimes \pi^{-1} e_1$, then we have

$$\begin{aligned} \text{Col}_1(x) &= x_2 - \varphi(x_1) \\ \text{Col}_2(x) &= qx_1 + \varphi(x_1) \end{aligned}$$

where we have replaced Col_2 by $-\text{Col}_2$ for simplicity.

Proposition 5.17. *The map $\text{Col}_1 : \mathbb{D}(T)^{\psi=1} \rightarrow (\mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$ is surjective.*

Proof. We first show that $(\pi \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0} \subset \text{Im}(\text{Col}_1)$. If $y \in (\pi \mathbb{A}_{\mathbb{Q}_p}^+)^{\psi=0}$, then the series $\sum_{i \geq 1} (-1)^i \frac{\varphi^{2i-1}(y)}{q \dots \varphi^{2i-2}(q)}$ and $\sum_{i \geq 0} (-1)^i \frac{\varphi^{2i}(y)}{\varphi(q) \dots \varphi^{2i-1}(q)}$ converge in $\mathbb{A}_{\mathbb{Q}_p}^+$ to elements x_1 and x_2 , respectively, and it is easy to see that $\psi(qx_2) = -x_1$ and $\psi(x_1) = x_2$. It follows that if we let $x = x_1 \log^-(1 + \pi) \nu_1 + x_2 \log^+(1 + \pi) \nu_2$, then $x \in \mathbb{D}(T)^{\psi=1}$, and moreover $\text{Col}_1(x) = x_2 - \varphi(x_1) = y$.

In order to prove surjectivity of Col_1 , it is hence sufficient to show that there exists $y \in \text{Im}(\text{Col}_1)$ with $y \equiv 1 \pmod{\pi}$. Let $y \in \mathbb{A}_{\mathbb{Q}_p}^{+, \psi=0}$ such that $\pi \mid ((1+\pi)^p + y)$. As above, the series $\sum_{i \geq 1} (-1)^i \frac{\varphi^{2i-1}(y) + \varphi^{2i}(1+\pi)}{q \dots \varphi^{2i-2}(q)}$ and $\sum_{i \geq 0} (-1)^i \frac{\varphi^{2i}(y) + \varphi^{2i+1}(1+\pi)}{\varphi(q) \dots \varphi^{2i-1}(q)}$ converge in $\mathbb{A}_{\mathbb{Q}_p}^+$. Let

$$z_1 = \frac{1}{2} \left((1+\pi) + \sum_{i \geq 1} (-1)^i \frac{\varphi^{2i-1}(y) + \varphi^{2i}(1+\pi)}{q \dots \varphi^{2i-2}(q)} \right),$$

$$z_2 = \frac{1}{2} \left(-\psi(q(1+\pi)) + \sum_{i \geq 0} (-1)^i \frac{\varphi^{2i}(y) + \varphi^{2i+1}(1+\pi)}{\varphi(q) \dots \varphi^{2i-1}(q)} \right).$$

It is easy to see that $\psi^2(q(1+\pi)) = 0$, so $\psi(qz_1) = -z_2$ and $\psi(z_2) = z_1$. It follows that if we let $x = z_1 \log^-(1+\pi)\nu_1 + z_2 \log^+(1+\pi)\nu_2$, then $x \in \mathbb{D}(T)^{\psi=1}$, and moreover

$$\text{Col}_1(x) = z_2 - \varphi(z_1) = 1 \pmod{\pi}.$$

□

Corollary 5.18. *The map $\text{Col}_1 : \mathbb{D}(T)^{\psi=1} \rightarrow \Lambda(G_\infty)$ is surjective.*

Proof. By Proposition 5.17, there exists $x \in \mathbb{N}(T)^{\psi=1}$ such that $\text{Col}_1(x) = 1+\pi$. The result therefore follows by precisely the same argument as in the proof of Theorem 4.28. □

Proposition 5.19. *The image of $\text{Col}_2 : \mathbb{D}(T)^{\psi=1} \rightarrow (\mathbb{A}_{\mathbb{Q}_p}^{+, \psi=0})^{\psi=0}$ is equal to $(\mathbb{A}_{\mathbb{Q}_p}^{+, \psi=0})^\Delta + \varphi(\pi)\mathbb{A}_{\mathbb{Q}_p}^{+, \psi=0}$.*

Proof. A similar argument to the one in the proof of Proposition 4.6 shows that $\varphi(\pi)\mathbb{A}_{\mathbb{Q}_p}^{+, \psi=0} \subset \text{Im}(\text{Col}_2)$. In [Fon90], Fontaine shows that $(\mathbb{A}_{\mathbb{Q}_p}^+)^{\Delta} = \mathbb{Z}_p[[\pi_0]]$, where $\pi_0 = -p + \sum_{a \in \mathbb{F}_p} [\varepsilon]^{[a]}$. Note that $\theta(\pi_0) = 0$ and $\theta \circ \varphi^{-1}(\pi_0) = -p$, so $\pi_0 = -p + \alpha q$ for some $\alpha \in \mathbb{A}_{\mathbb{Q}_p}^+$ satisfying $\alpha \equiv 1 \pmod{\pi}$. Now $\{[\varepsilon]^{[a]}\}_{a \in \mathbb{F}_p^\times}$ is a basis for $\mathbb{A}_{\mathbb{Q}_p}^+$ over $\varphi(\mathbb{A}_{\mathbb{Q}_p}^+)$, so $\psi(\pi_0) = 1 - p$, and hence $\pi_0 + p - 1 \in \mathbb{A}_{\mathbb{Q}_p}^{+, \psi=0}$. In order to prove that $(\mathbb{A}_{\mathbb{Q}_p}^{+, \psi=0})^\Delta \subset \text{Im}(\text{Col}_2)$, it is therefore sufficient to prove the following results:

- (1) $\pi_0 + p - 1 \in \text{Im}(\text{Col}_2)$;
- (2) If $y \in (\mathbb{A}_{\mathbb{Q}_p}^{+, \psi=0})^\Delta$, then $y = c(\pi_0 + p - 1) \pmod{\varphi(\pi)}$ for some $c \in \mathbb{Z}_p$.

Proof of claim 1. Note that since $\pi_0 + p - 1 = -1 + q \pmod{\varphi(\pi)}$, (a) is equivalent to showing that there exists $y \in \text{Im}(\text{Col}_2)$ such that $y = -1 + q \pmod{\varphi(\pi)}$. If $i(x) = x_1 \log^-(1+\pi)\nu_1 + x_2 \log^+(1+\pi)\nu_2$ for some $x \in \mathbb{D}(T)^{\psi=1}$, then $\text{Col}_2(x) = qx_1 + \varphi(x_2)$. As shown in Lemma 5.12, we have $\theta(x_1) = -\theta(x_2)$, so

$$\text{Col}_2(x) \equiv \theta(x_2)(1 - q) \pmod{\varphi(\pi)}.$$

Suppose now that $\theta(x_2) = 0$ for all $x \in \mathbb{D}(T)^{\psi=1}$. Then, the fact that $\text{Col}_1(x) \equiv \theta(x_2) - \theta(x_1) \pmod{\pi}$ implies that $\text{Col}_1(x) \in \pi\mathbb{A}_{\mathbb{Q}_p}^+$ for all $x \in \mathbb{D}(T)^{\psi=1}$, which contradicts the surjectivity of Col_1 . □

Proof of claim 2. We will show that if $y \in (\mathbb{A}_{\mathbb{Q}_p}^{+, \psi=0})^\Delta$, then

$$(56) \quad y = c(-1 + q) \pmod{\varphi(\pi)}$$

for some $c \in \mathbb{Z}_p$. Write $y = f(\pi_0) = g(\pi)$. In order to show (56), it is sufficient to prove that $g(0) = -(p-1)g(\zeta_p - 1)$. The condition that $y \in \ker(\psi)$ translates as

$$\frac{1}{p} \sum_{\xi^p=1} f\left(-p + \sum_{a \in \mathbb{F}_p} \xi^a (\pi + 1)^{[a]}\right) = 0.$$

Evaluating this condition at $\pi = 0$ shows that $f(0) + (p-1)f(-p) = 0$. By definition, we have $\pi_0 = -p + \sum_{a \in \mathbb{F}_p} (\pi + 1)^{[a]}$, so $g(0) = f(0)$ and $g(\zeta_p - 1) = f(-p)$, which finishes the proof. □

This completes the proof of proposition 5.19. □

Let $\eta : \Delta \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times$ be a tame character. For a $\Lambda(G_\infty)$ -module A , denote by A^η the $\Lambda(G_\infty)$ -submodule of A on which Δ acts via η . The following result is an immediate consequence of Proposition 5.19.

Corollary 5.20. *We have*

$$\text{Im}(\text{Col}_2)^\eta = \begin{cases} (\mathbb{A}_{\mathbb{Q}_p}^{+, \psi=0})^\Delta & \text{if } \eta = 1 \\ (\varphi(\pi) \mathbb{A}_{\mathbb{Q}_p}^{+, \psi=0})^\eta & \text{otherwise} \end{cases}$$

We can translate Proposition 5.19 and Corollary 5.20 into a statement about $\text{Im}(\underline{\text{Col}}_2)$.

Proposition 5.21. *The image of $\underline{\text{Col}}_2 : \mathbb{D}(T)^{\psi=1} \rightarrow \Lambda(G_\infty)$ is equal to $(\sum_{i=1}^{p-1} \delta^i) \Lambda(G_\infty) + (\gamma - 1) \Lambda(G_\infty)$.*

Proof. Let $y_2 = \varphi(\pi)(1 + \pi) \in \text{Im}(\text{Col}_2)$. As shown in the proof of Proposition 5.19, $y = (0, y_2) \in \text{Im}(\text{Col})$; more precisely, there exists $x \in \mathbb{N}(T)^{\psi=1}$ such that $\text{Col}(x) = y$. Applying the algorithm for \mathfrak{J} (see Section 4.4) to y shows that $\underline{\text{Col}}_2(x) = (\gamma - 1) \pmod{(p, (\gamma - 1)^2)}$, so the $\Lambda(G_\infty)$ -submodule of $\Lambda(G_\infty)$ generated by $\underline{\text{Col}}_2(x)$ is equal to the ideal generated by $(\gamma - 1)$.

Furthermore, $y'_2 \sum_{i=1}^{p-1} (\pi + 1)^i \in \text{Im}(\text{Col}_2)$ by Proposition 5.19, and every $y \in \text{Im}(\text{Col}_2)$ is congruent to a scalar multiple of $y'_2 \pmod{\varphi(\pi)}$. If $x' \in \mathbb{N}(T)^{\psi=1}$ satisfies $\text{Col}_2(x') = y'_2$, then again the algorithm for \mathfrak{J} implies that $\underline{\text{Col}}_2(x) = \sum_{i=1}^{p-1} \delta^i \pmod{(\gamma - 1)}$. This finishes the proof. \square

Corollary 5.22. *We have*

$$\text{Im}(\underline{\text{Col}}_2)^\eta = \begin{cases} \Lambda(G_\infty)^\Delta = (\sum_{i=1}^{p-1} \delta^i) \Lambda(G_\infty) & \text{if } \eta = 1 \\ ((\gamma - 1) \Lambda(G_\infty))^\eta & \text{otherwise} \end{cases}$$

Note that the results of Corollaries 5.18 and 5.22 are equivalent to Theorem 6.2 in [Kob03].

5.4. The case $k = 2$. In this section we consider the case of modular forms which have weight 2 and are non-ordinary at p . For modular forms with trivial character and coefficients in \mathbb{Q} (hence corresponding to elliptic curves), but with $a_p \neq 0$, this case was studied in detail by Sprung.

5.4.1. Coleman maps via the Perrin-Riou pairing. We first review Sprung's construction of the Coleman maps for elliptic curves over \mathbb{Q} with $p \mid a_p$ but $a_p \neq 0$, and explain how we can rewrite these Coleman maps using Perrin-Riou's pairing.

Let f be a modular form as in Section 3.3 with $k = 2$. Define for $n \geq 1$

$$\begin{pmatrix} \Theta_n^1 & \Upsilon_n^1 \\ \Theta_n^0 & \Upsilon_n^0 \end{pmatrix} = \begin{pmatrix} 0 & \Phi_n(\gamma) \\ -1 & a_p \end{pmatrix} \cdots \begin{pmatrix} 0 & \Phi_1(\gamma) \\ -1 & a_p \end{pmatrix} \in M_2(\mathcal{H}(G_\infty)).$$

Then, we have:

Lemma 5.23. *Let $i \in \mathbb{Z}$ and write*

$$A_n^i = \begin{pmatrix} 0 & p \\ -1 & a_p \end{pmatrix}^i \begin{pmatrix} \Theta_n^1 & \Upsilon_n^1 \\ \Theta_n^0 & \Upsilon_n^0 \end{pmatrix}.$$

Then, A_n^{i-n} converges in $M_2(\mathcal{H}(G_\infty))$ as $n \rightarrow \infty$ for a fixed i . Write A_∞^i for the limit, then all entries of A_∞^i are $O(\log_p^{1/2})$. Moreover, if η is a character on G_∞ which factors through G_n but not G_{n-1} , then $\eta(A_\infty^i) = \eta(A_m^{i-m})$ for any $m \geq n - 1$.

Proof. [Spr09, Lemma 3.21] \square

Proposition 5.24. *For any $\mathbf{z} \in H_{\text{Iw}}^1(V_{\bar{f}}(1))$ and $0 \neq \omega \in \mathbb{D}_{\text{cris}}^1(V_f)$, the entries of the row vector*

$$\left(\frac{1}{p} \mathcal{L}_{1, (1+\pi) \otimes \varphi(\omega)}(z) \quad -\mathcal{L}_{1, (1+\pi) \otimes \omega}(z) \right) A_\infty^{-1}$$

are both divisible by $\log_p(\gamma)/(\gamma - 1)$.

Proof. For $n \in \mathbb{Z}$, write $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ where α and β are the roots of $X^2 - a_p X + p$. Then, $\varphi^n = u_n \varphi - p u_{n-1}$ and

$$\begin{pmatrix} 0 & p \\ -1 & a_p \end{pmatrix}^n = \begin{pmatrix} -p u_{n-1} & p u_n \\ -u_n & u_{n+1} \end{pmatrix}.$$

Therefore, if $n > 1$ and η is a character of G_∞ which factors through G_n but not G_{n-1} (so $\eta(\gamma)$ is a primitive p^{n-1} -th root of unity), we have

$$\eta(A_\infty^{-1}) = \begin{pmatrix} -p u_{-n-1} & p u_{-n} \\ -u_{-n} & u_{-n+1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & a_p \end{pmatrix} \eta \begin{pmatrix} \Theta_{n-2}^1 & \Upsilon_{n-2}^1 \\ \Theta_{n-2}^0 & \Upsilon_{n-2}^0 \end{pmatrix}$$

where the last matrix is the identity if $n = 2$.

By [Lei09, Section 1.1.4], we have

$$\eta(\mathcal{L}_{1,(1+\pi) \otimes v}(\mathbf{z})) = \frac{1}{\tau(\eta^{-1})} \sum_{\sigma \in G_n} \eta^{-1}(\sigma) [\varphi^{-n}(v), \exp_{n,1}^*(z_n^\sigma)]_n$$

for any $v \in \mathbb{D}_{\text{cris}}(V_f)$ and $z \in H_{\text{Iw}}^1(V_{\bar{f}}(1))$. Hence, if $\omega \in \mathbb{D}_{\text{cris}}^1(V_f)$, then

$$\eta \left(\left(\frac{1}{p} \mathcal{L}_{1,(1+\pi) \otimes \varphi(\omega)}(z) \quad -\mathcal{L}_{1,(1+\pi) \otimes \omega}(z) \right) A_\infty^{-1} \right) = 0$$

because

$$\left(\frac{1}{p} u_{-n+1} \quad -u_{-n} \right) \begin{pmatrix} -p u_{-n-1} & p u_{-n} \\ -u_{-n} & u_{-n+1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & a_p \end{pmatrix} = 0,$$

which implies that

$$\left(\frac{1}{p} \varphi^{-n+1}(\omega) \quad -\varphi^{-n}(\omega) \right) \begin{pmatrix} -p u_{-n-1} & p u_{-n} \\ -u_{-n} & u_{-n+1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & a_p \end{pmatrix} \equiv 0 \pmod{\mathbb{D}_{\text{cris}}^1(V_f)}.$$

□

By [PR94], the image of $\mathcal{L}_{1,(1+\pi) \otimes v}$ is $O(\log_p^{1/2})$ for any $v \in \mathbb{D}_{\text{cris}}(V_f)$, so we obtain two Coleman maps:

Definition 5.25. Fix a non-zero element $\omega \in \mathbb{D}_{\text{cris}}^1(V_f)$. For $* = \vartheta, v$ and $z \in H_{\text{Iw}}^1(V_{\bar{f}}(1))$, $\text{Col}^*(z) \in \Lambda_E(G_\infty)$ is defined by

$$(57) \quad (\text{Col}^\vartheta(z) \quad \text{Col}^v(z)) \cdot \log_p(\gamma)/p(\gamma - 1) = \left(\frac{1}{p} \mathcal{L}_{(1+\pi) \otimes \varphi(\omega)}(z) \quad -\mathcal{L}_{(1+\pi) \otimes \omega}(z) \right) A_\infty^{-1}.$$

In particular, we can define two p -adic L -functions

$$\tilde{L}_p^* = \text{Col}^*(\mathbf{z}^{\text{Kato}}) \in \Lambda_E(G_\infty)$$

where \mathbf{z}^{Kato} is the localization of the Kato zeta element and $* = \vartheta, v$.

Remark 5.26. The results above hold for any modular forms with $k = 2$, $p \nmid N$ and $v_p(a_p) \geq 1/2$. This setting is slightly more general than that in [Spr09].

5.4.2. Compatibility of Coleman maps. Since condition (C) holds and $k = 2$, with respect to the canonical basis of $\mathbb{N}(V_f)$ given above, P is simply

$$(58) \quad \begin{pmatrix} 0 & -1 \\ q & a_p \end{pmatrix}.$$

Write B_∞^i (respectively B_n^i) for the matrix obtained from A_∞^i (respectively A_n^i) by replacing $\Phi_m(\gamma)$ by $\varphi^{m-1}(q)$ for all m . Then, we have:

Lemma 5.27. Under the notation above, $M' = B_\infty^0$.

Proof. By (58), $(B_n^{-n})^T = P\varphi(P) \cdots \varphi^{n-1}(P)A_\varphi^{-n}$. For $\gamma \in G_\infty$, we write $G_\gamma^{(n)} = (B_n^{-n})^T \cdot \gamma((B_n^{-n})^T)^{-1}$. Then,

$$P \cdot \varphi(G_\gamma^{(n)}) \cdot \gamma(P)^{-1} = G_\gamma^{(n+1)}.$$

Hence, if we write G_γ for the limit of $G_\gamma^{(n)}$ as $n \rightarrow \infty$, then

$$P \cdot \varphi(G_\gamma) \cdot \gamma(P)^{-1} = G_\gamma,$$

It is easy to check that G_γ satisfies $G_{\gamma_1\gamma_2} = G_{\gamma_1} \cdot \gamma_1(G_{\gamma_2})$ for any $\gamma_1, \gamma_2 \in G_\infty$. Hence, we recover the action of G_∞ on the Wach module $\mathbb{N}(V_f)$. In other words, G_γ is the matrix of γ with respect to the basis n_1, n_2 chosen in [BLZ04]. Since $G_\gamma = (B_\infty^0)^T \cdot \gamma((B_\infty^0)^T)^{-1}$ and $G_\gamma|_{\pi=0} = I$, we have

$$B_\infty^0 \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \left((E \otimes \mathbb{B}_{\text{rig}, \mathbb{Q}_p}^+) \otimes \mathbb{N}(V_f) \right)^{G_\infty} = \mathbb{D}_{\text{cris}}(V_f)$$

and $M' = B_\infty^0$. \square

We write $A^c = \det(A)A^{-1}$ if A is an invertible matrix, then we have:

Corollary 5.28. *The matrix M can be obtained from $(A_\infty^{-1})^c$ by replacing Φ_m by $\varphi(q)^m$.*

Proof. Recall that

$$M = \frac{t}{\pi q} P^T (M')^{-1} = \frac{t}{\pi q} \varphi(M'^{-1}) A_\varphi^T.$$

By Lemma 5.27, $\det(M') = \det(B_\infty^0) = \prod_{n \geq 0} \frac{\varphi^n(q)}{p} = t/\pi$. But $\det A_{vp} = p$ and $A_\infty^{i+1} = A_\varphi^T A_\infty^i$ for all i . Hence, we have

$$M = \varphi((A_\varphi^T)^{-1} B_\infty^0)^c = \varphi(B_\infty^{-1})^c$$

and we are done. \square

On setting $\nu_1 = -\omega$ in (57), (32) implies that

$$(59) \quad (\underline{\text{Col}}_1 \quad \underline{\text{Col}}_2) \underline{M} A_\infty^{-1} = (\text{Col}^\vartheta \circ h_{\text{Iw}}^1 \quad \text{Col}^v \circ h_{\text{Iw}}^1) \log_p(\gamma)/p(\gamma - 1).$$

By [Spr09],

$$\text{Im}(\text{Col}^\vartheta) = \text{Im}(\text{Col}^v) = \Lambda_E(G_\infty)$$

and (59) implies that the matrix $\underline{M} A_\infty^{-1}$ defines a $\Lambda_E(G_\infty)$ -linear map from $\Lambda_E(G_\infty)^{\oplus 2}$ onto $(\log_p(\gamma)/p(\gamma - 1)\Lambda_E(G_\infty))^{\oplus 2}$. Hence, there exists $A \in GL_2(\Lambda_E(G_\infty))$, $\underline{M} A_\infty^{-1} = [\log_p(\gamma)/p(\gamma - 1)]A$. This implies

$$(\underline{\text{Col}}_1 \quad \underline{\text{Col}}_2) A = (\text{Col}^\vartheta \circ h_{\text{Iw}}^1 \quad \text{Col}^v \circ h_{\text{Iw}}^1).$$

We also see that \underline{M} and $(A_\infty^{-1})^c$ agree up to an element in $GL_2(\Lambda_E(G_\infty))$ which is a generalization of Proposition 5.10 because of the description of M in Corollary 5.28.

6. MAIN CONJECTURES

6.1. Kato's main conjecture. In general, if V is a p -adic representation of $G_\mathbb{Q}$ unramified outside a finite set of primes, and T is a \mathbb{Z}_p -lattice in V stable under $G_\mathbb{Q}$, we write

$$\begin{aligned} \mathbb{H}^i(T) &= \varprojlim_n H_{\text{ét}}^i \left(\text{Spec } \mathbb{Z}[\zeta_{p^n}, \frac{1}{p}], j_* T \right), \\ \mathbb{H}^i(V) &= \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{H}^i(T). \end{aligned}$$

for $i = 1, 2$; see [Kat04, §§8.2 & 12.2]. Here j is the natural map $\text{Spec } \mathbb{Q}(\zeta_{p^n}) \rightarrow \text{Spec } \mathbb{Z}[\zeta_{p^n}, \frac{1}{p}]$. Note that $\mathbb{H}^i(V)$ is independent of the choice of lattice T .

We now continue under the notation of Section 3.3 and Section 3.6. Fix a uniformizer ϖ of \mathcal{O}_E . Let $\mathbb{Z}(T_f) \subset \mathbb{H}^1(T_f)$ denote the $\Lambda_{\mathcal{O}_E}(G_\infty)$ -module generated by the Kato zeta elements as defined in [Kat04, Theorem 12.5] and write $\mathbb{Z}(V_f) = \mathbb{Z}(T_f) \otimes \mathbb{Q}$. The following assumption will be needed for some of the results below.

- **Assumption (E):** there exists a basis of T_f for which the image of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_\infty)$ in $\text{GL}_2(\mathcal{O}_E)$ contains $\text{SL}_2(\mathbb{Z}_p)$.

Theorem 6.1 ([Kat04, theorem 12.5]). *Let $\eta : \Delta \rightarrow \mathbb{Z}_p^\times$ be a character, then:*

- $\mathbb{H}^2(T_f)$ is a torsion $\Lambda_{\mathcal{O}_E}(G_\infty)$ -module.
- $\mathbb{H}^1(T_f)$ is a torsion free $\Lambda_{\mathcal{O}_E}(G_\infty)$ -module and $\mathbb{H}^1(V_f)$ is a free $\Lambda_E(G_\infty)$ -module of rank 1.
- The quotient $\mathbb{H}^1(V_f)/\mathbb{Z}(V_f)$ is a torsion $\Lambda_E(G_\infty)$ -module.
- $\text{Char}_{\Lambda_E(\Gamma)}(\mathbb{H}^1(V_f)^\eta/\mathbb{Z}(V_f)^\eta) \subset \text{Char}_{\Lambda_E(\Gamma)}(\mathbb{H}^2(V_f)^\eta)$.
- If assumption (E) holds, then $\mathbb{Z}(T_f) \subset \mathbb{H}^1(T_f)$. Moreover, $\mathbb{H}^1(T_f)$ is a free $\Lambda_{\mathcal{O}_E}(G_\infty)$ -module of rank 1 and

$$\text{Char}_{\Lambda_{\mathcal{O}_E}(\Gamma)}(\mathbb{H}^1(T_f)^\eta/\mathbb{Z}(T_f)^\eta) \subset \text{Char}_{\Lambda_{\mathcal{O}_E}(\Gamma)}(\mathbb{H}^2(T_f)^\eta).$$

Kato's main conjecture states that:

Conjecture 6.2. *Let $\eta : \Delta \rightarrow \mathbb{Z}_p^\times$ be a character, then $\mathbb{Z}(T_f)^\eta \subset \mathbb{H}^1(T_f)^\eta$ and*

$$\text{Char}_{\Lambda_{\mathcal{O}_E}(\Gamma)}(\mathbb{H}^1(T_f)^\eta/\mathbb{Z}(T_f)^\eta) = \text{Char}_{\Lambda_{\mathcal{O}_E}(\Gamma)}(\mathbb{H}^2(T_f)^\eta).$$

Remark 6.3. *The above formulation of the conjecture can be found in [Kob03, §5]; it is more convenient for our purposes than the original formulation (Conjecture 12.10 of [Kat04]).*

6.2. Reformulation of Kato's main conjecture. Let K be a number field. The p -Selmer group of f over K is defined by

$$\text{Sel}_p(f/K) = \ker \left(H^1(K, V_f/T_f(1)) \rightarrow \prod_{\nu} \frac{H^1(K_\nu, V_f/T_f(1))}{H_f^1(K_\nu, V_f/T_f(1))} \right)$$

where ν runs through all the places of K .

We choose a “good basis” ν_1, ν_2 of $\mathbb{D}_{\text{cris}}(V_{\bar{f}})$ in the sense of subsection 3.3. Lemma 3.15 shows that we may find a lift n_1, n_2 of this to a basis of $\mathbb{N}(V_{\bar{f}})$ such that $(1 + \pi)\varphi(\pi^{1-k}n_1 \otimes e_{k-1}), (1 + \pi)\varphi(\pi^{1-k}n_2 \otimes e_{k-1})$ is a Λ_E -basis of $\mathbb{N}(V_{\bar{f}}(k-1))$. We choose such basis (n_1, n_2) .

With respect to this basis, we write $H_f^1(\mathbb{Q}_{p,n}, V_f/T_f(1))^i$ for the annihilator of the projection of $\ker(\underline{\text{Col}}_i)$ in $H^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(k-1))$ under the pairing

$$H^1(\mathbb{Q}_{p,n}, T_{\bar{f}}(k-1)) \times H^1(\mathbb{Q}_{p,n}, V_f/T_f(1)) \rightarrow E/\mathcal{O}_E.$$

This enables us to make the following definition:

Definition 6.4.

$$\begin{aligned} \text{Sel}_p^i(f/\mathbb{Q}(\mu_{p^n})) &= \ker \left(\text{Sel}_p(f/\mathbb{Q}(\mu_{p^n})) \rightarrow \frac{H^1(\mathbb{Q}_{p,n}, V_f/T_f(1))}{H_f^1(\mathbb{Q}_{p,n}, V_f/T_f(1))^i} \right) \\ \text{Sel}_p^i(f/\mathbb{Q}_\infty) &= \varinjlim_n \text{Sel}_p^i(f/\mathbb{Q}(\mu_{p^n})). \end{aligned}$$

By the Poitou-Tate exact sequence (see [Kob03, Section 7] and [Lei09, Section 4]), we have

$$(60) \quad \mathbb{H}^1(T_{\bar{f}}(k-1)) \rightarrow \text{Im}(\underline{\text{Col}}_i) \rightarrow \text{Sel}_p^i(f/\mathbb{Q}_\infty)^\vee \rightarrow \mathbb{H}^2(T_{\bar{f}}(k-1)) \rightarrow 0$$

where $(\cdot)^\vee$ denotes the Pontryagin dual.

Theorem 6.5. *Under assumption (A) (if f is supersingular at p) or assumption (A') (if f is ordinary at p), $\text{Sel}_p^i(f/\mathbb{Q}_\infty)$ is $\Lambda_{\mathcal{O}_E}(G_\infty)$ -cotorsion. Moreover, there exist some $n_i \geq 0$ such that*

$$\varpi^{n_i} \tilde{L}_{p,i}^\eta \in \text{Char}_{\Lambda_{\mathcal{O}_E}(\Gamma)}(\text{Sel}_p^i(f/\mathbb{Q}_\infty)^{\vee, \eta})$$

where η is any character on Δ when $i = 1$ and it is the trivial character when $i = 2$.

Proof. Assume f is supersingular at p . By Corollary 3.29, assumption (A) implies that $\tilde{L}_{p,i}^\eta \neq 0$. Hence, the cokernel of the first map in (60) is $\Lambda_{\mathcal{O}_E}(G_\infty)$ -torsion. But $\mathbb{H}^2(T_{\bar{f}}(k-1))$ is also $\Lambda_{\mathcal{O}_E}(G_\infty)$ -torsion by [Kat04], so $\text{Sel}_p^i(f/\mathbb{Q}_\infty)^\vee$ is $\Lambda_{\mathcal{O}_E}(G_\infty)$ -torsion, too.

As in [Kob03, Theorem 7.3], the first arrow of (60) is now injective and there exist $n \geq 0$ such that

$$(61) \quad 0 \rightarrow \mathbb{H}^1(T_{\bar{f}}(k-1))/\mathbb{Z}(T_{\bar{f}}(k-1)) \rightarrow \text{Im}(\underline{\text{Col}}_i)/(\varpi^{n_i} \tilde{L}_{p,i}) \rightarrow \text{Sel}_p^i(f/\mathbb{Q}_\infty)^\vee \rightarrow \mathbb{H}^2(T_{\bar{f}}(k-1)) \rightarrow 0.$$

Hence, the second part of the theorem follows from Theorem 6.1(d) on taking η -components. The proof for the ordinary case is analogous. \square

Corollary 6.6. *Let η be a character on Δ as above. If assumptions (A) (or (A')) depending on whether f is supersingular or ordinary at p) and (E) hold, then Kato's main conjecture is equivalent to*

$$\text{Char}_{\Lambda_{\mathcal{O}_E}(\Gamma)}(\text{Sel}_p^i(f/\mathbb{Q}_\infty)^{\vee,\eta}) = \text{Char}_{\Lambda_{\mathcal{O}_E}(\Gamma)}(\text{Im}(\underline{\text{Col}}_i)^\eta / (\tilde{L}_{p,i}^\eta)).$$

Proof. It follows immediately from (61). \square

Remark 6.7. *We do not assume that n_1, n_2 is an $\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}^+$ -basis for $\mathbb{N}(T_{\bar{f}})$. Hence $\text{Im}(\underline{\text{Col}}_i)$ need not be contained in $\Lambda_{\mathcal{O}_E}(G_\infty)$; but it is still clearly a $\Lambda_{\mathcal{O}_E}(G_\infty)$ -submodule of $\Lambda_E(G_\infty)$.*

By Theorem 4.28, if f is supersingular at p , then assumptions (B), (C) and (D) imply that $\text{Im}(\underline{\text{Col}}_1) = \Lambda_E(G_\infty)$. Therefore, we can reformulate Kato's main conjecture in the following form:

Corollary 6.8. *If f is supersingular at p and assumptions (A)-(D) all hold, then Kato's main conjecture (after tensoring by \mathbb{Q}) is equivalent to the assertion that $\text{Char}_{\Lambda_E(\Gamma)}(\text{Sel}_p^1(f/\mathbb{Q}_\infty)^{\vee,\eta})$ is generated by $\tilde{L}_{p,1}^\eta$.*

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